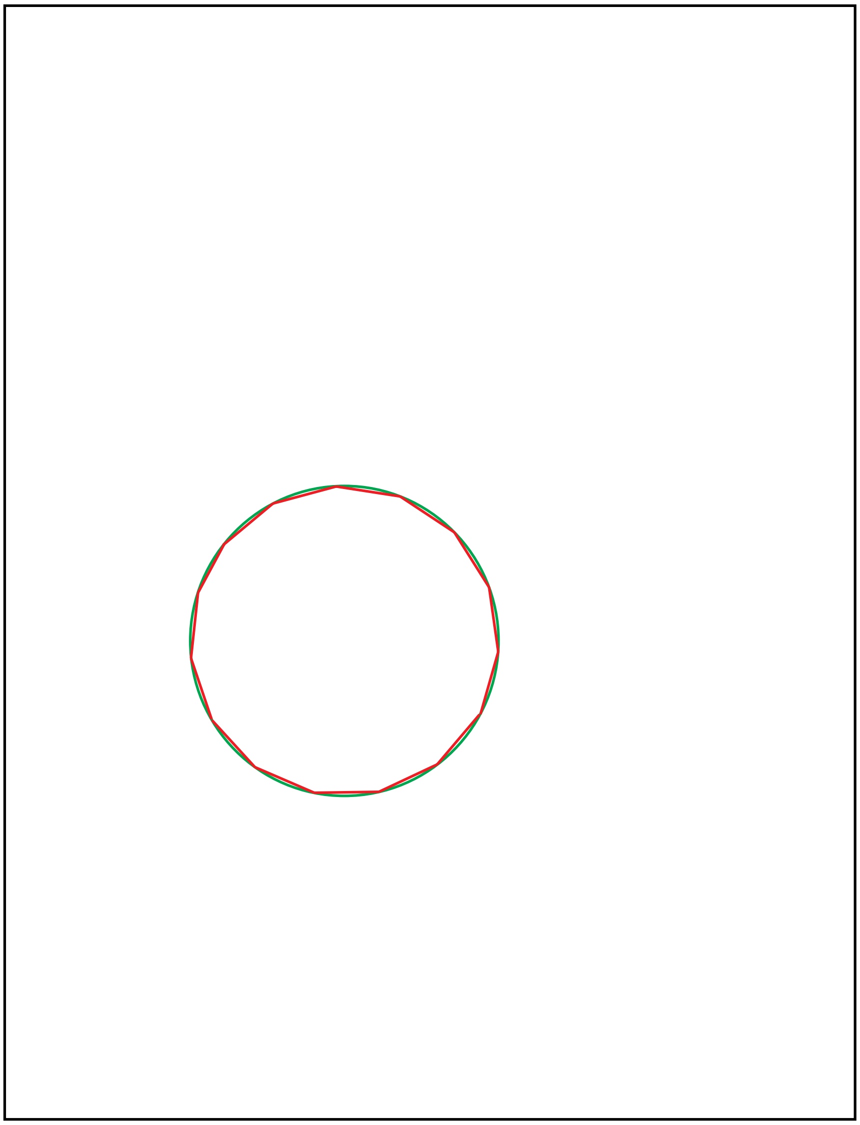
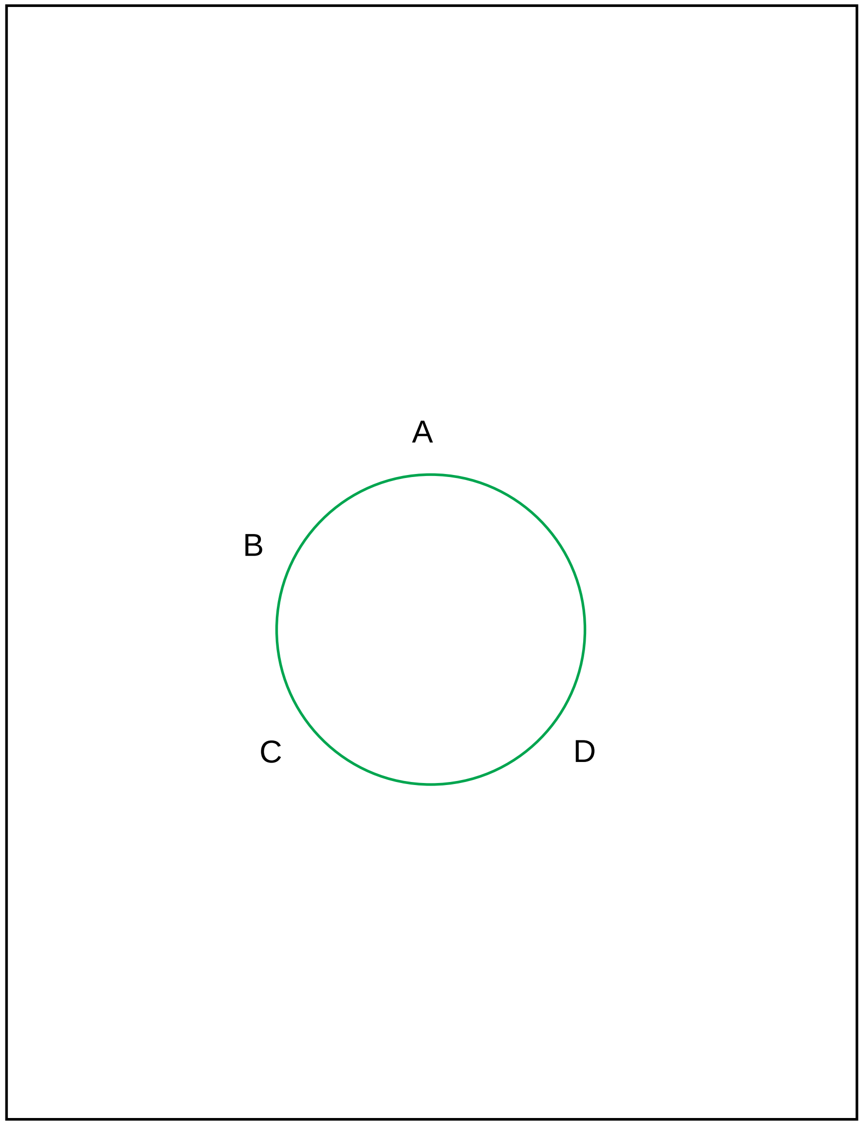
Construction 38: Book IV, Proposition 16

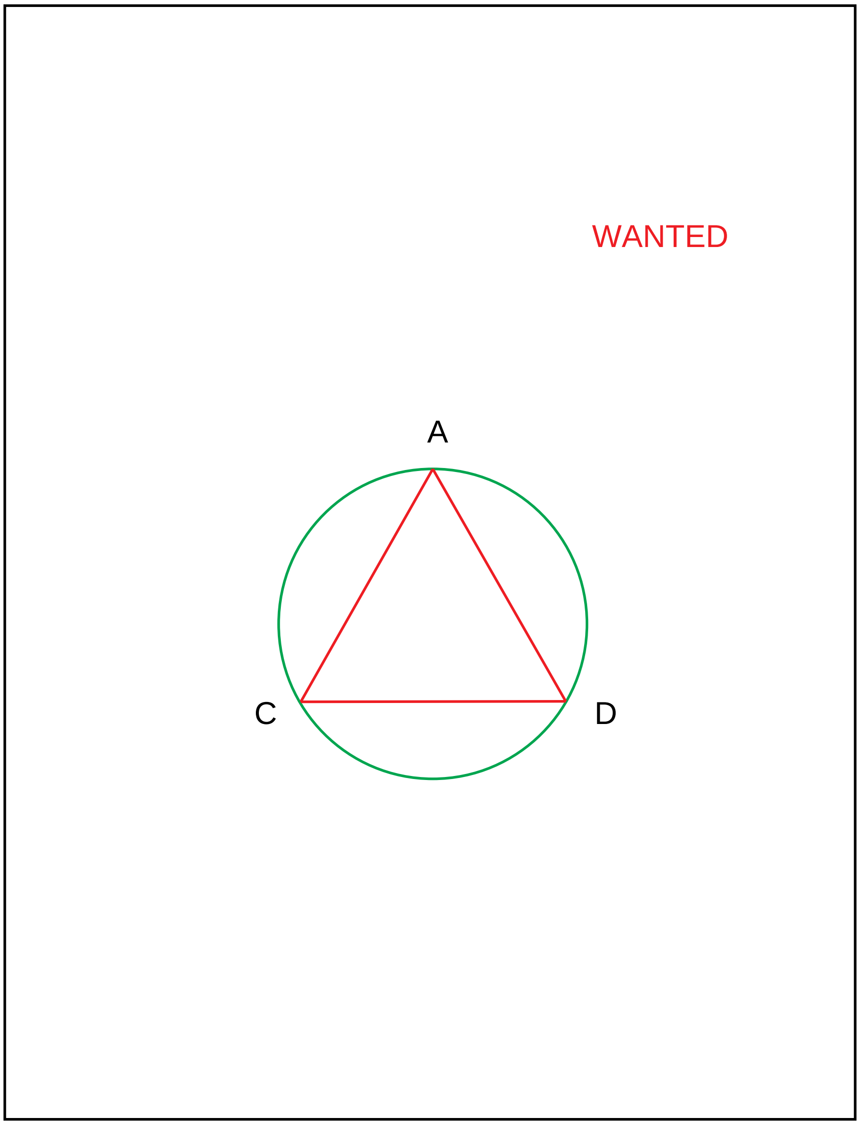
In *a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.*



IV.16:3. Let ABCD be the given circle;



IV.16:8. In the circle ABCD let there be inscribed a side AC of the equilateral triangle inscribed in it, ([IV.2])



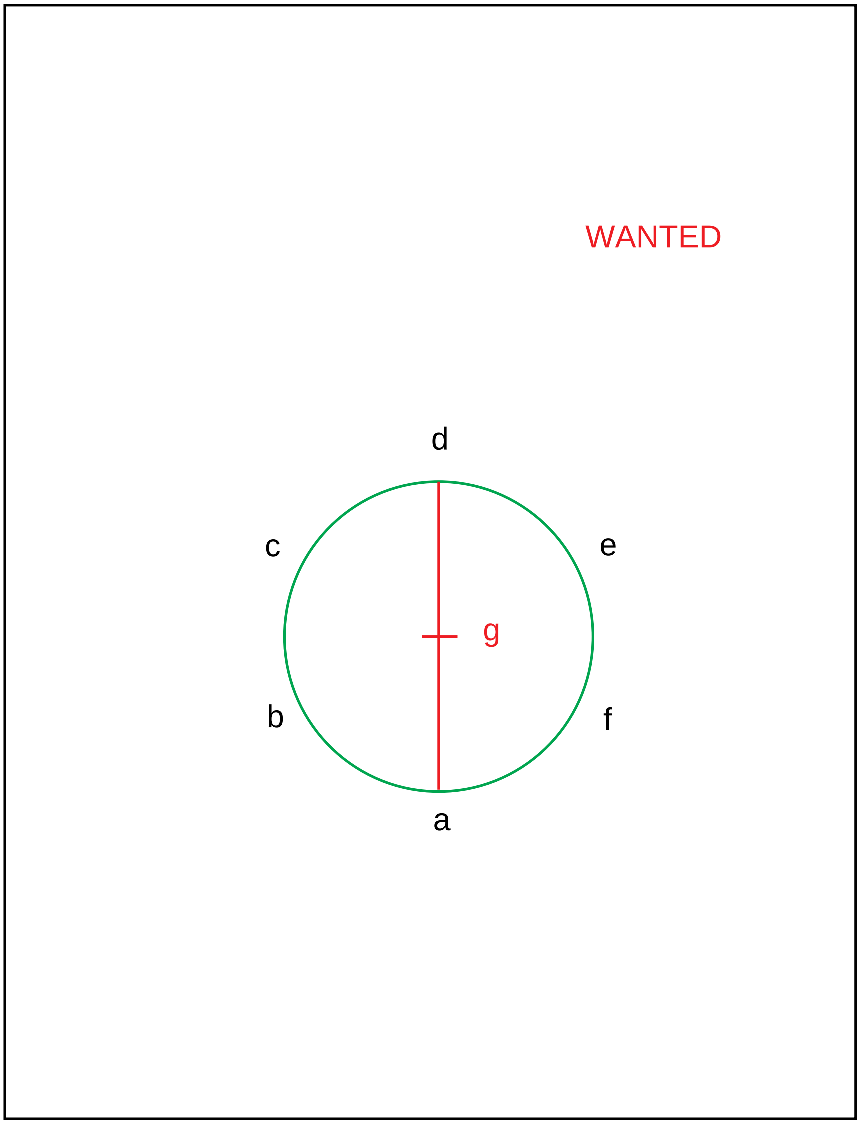
Rather than GOSUB IV.2 (22 steps) we prefer GOSUB IV.15 (11 steps) to construct a hexagon, then use 3 of its 6 vertices.

GOSUB IV.15. Relabel.

IV.15:3. Let abcdef be the given circle;

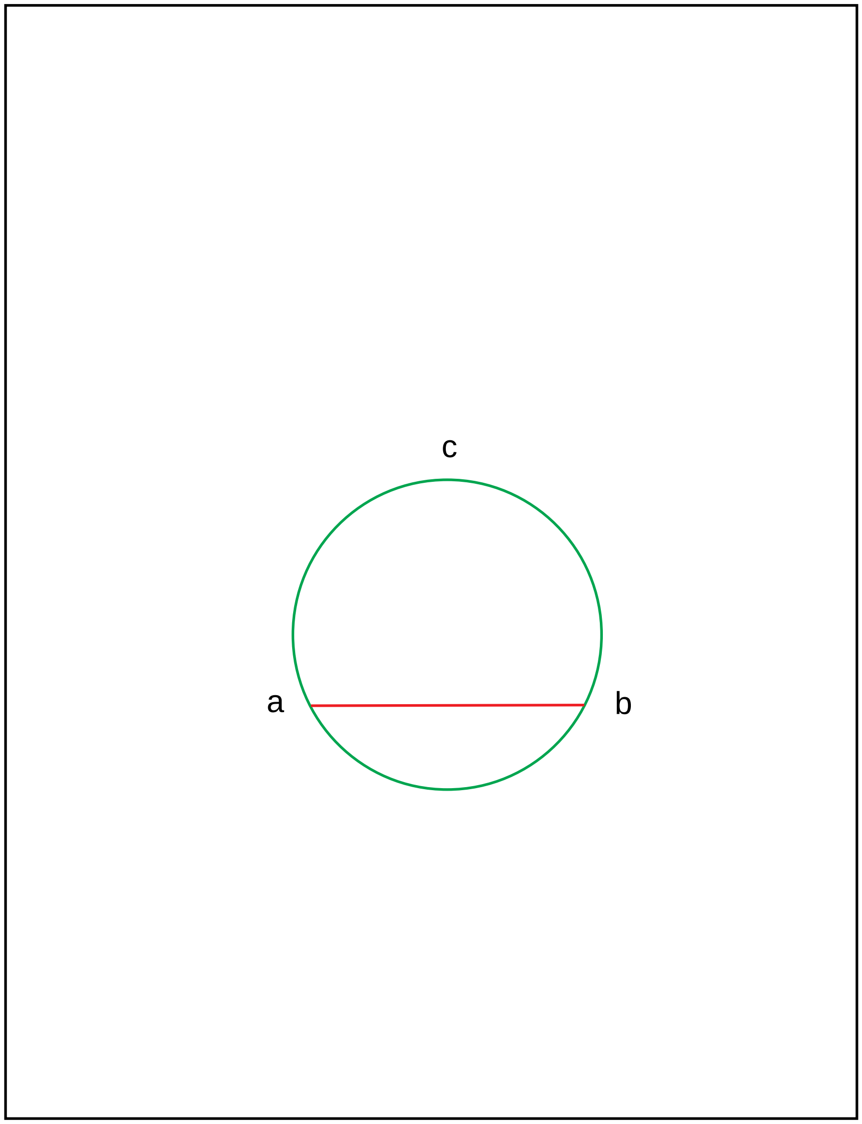
IV.15:6. Let the diameter ad of the circle abcdef be drawn; let the centre g of the circle be taken, ([III.1])

GOSUB III.1.
Relabel.



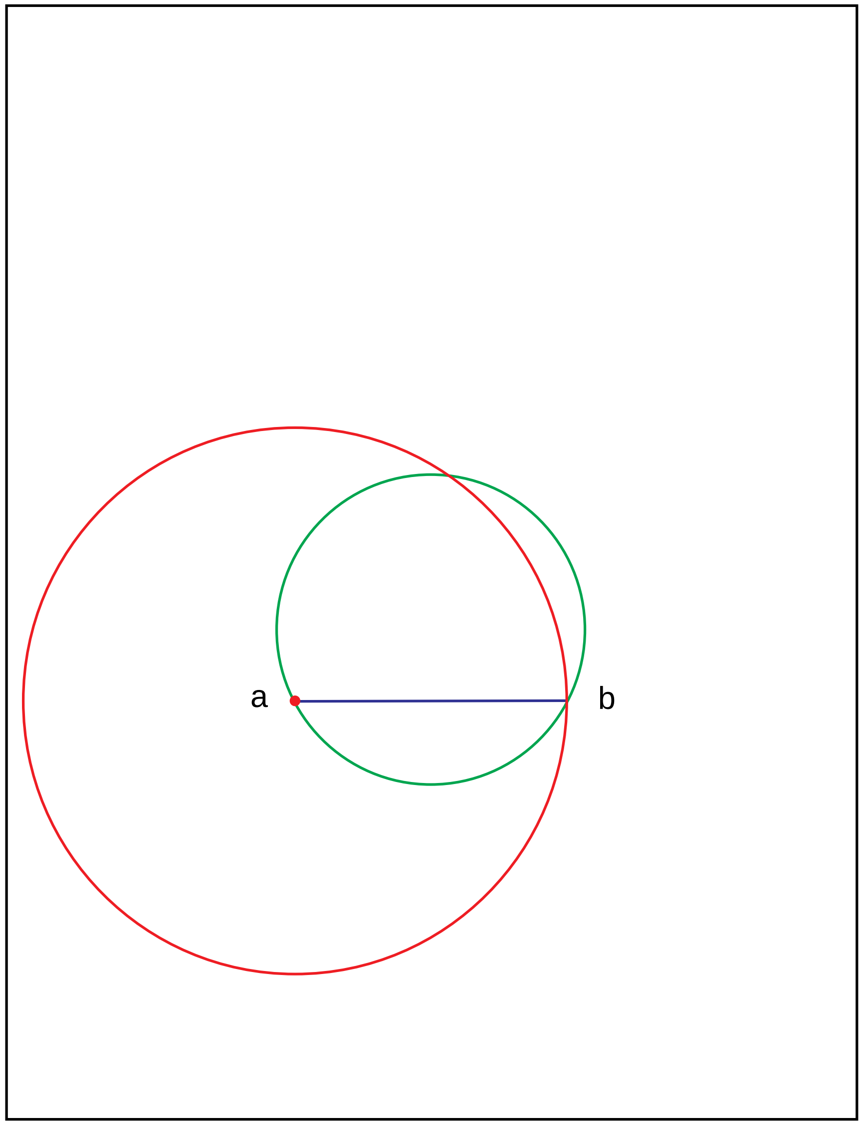
III.1:2. Let abc be the given circle;

III.1:4. Let a straight line ab be drawn through it at random,

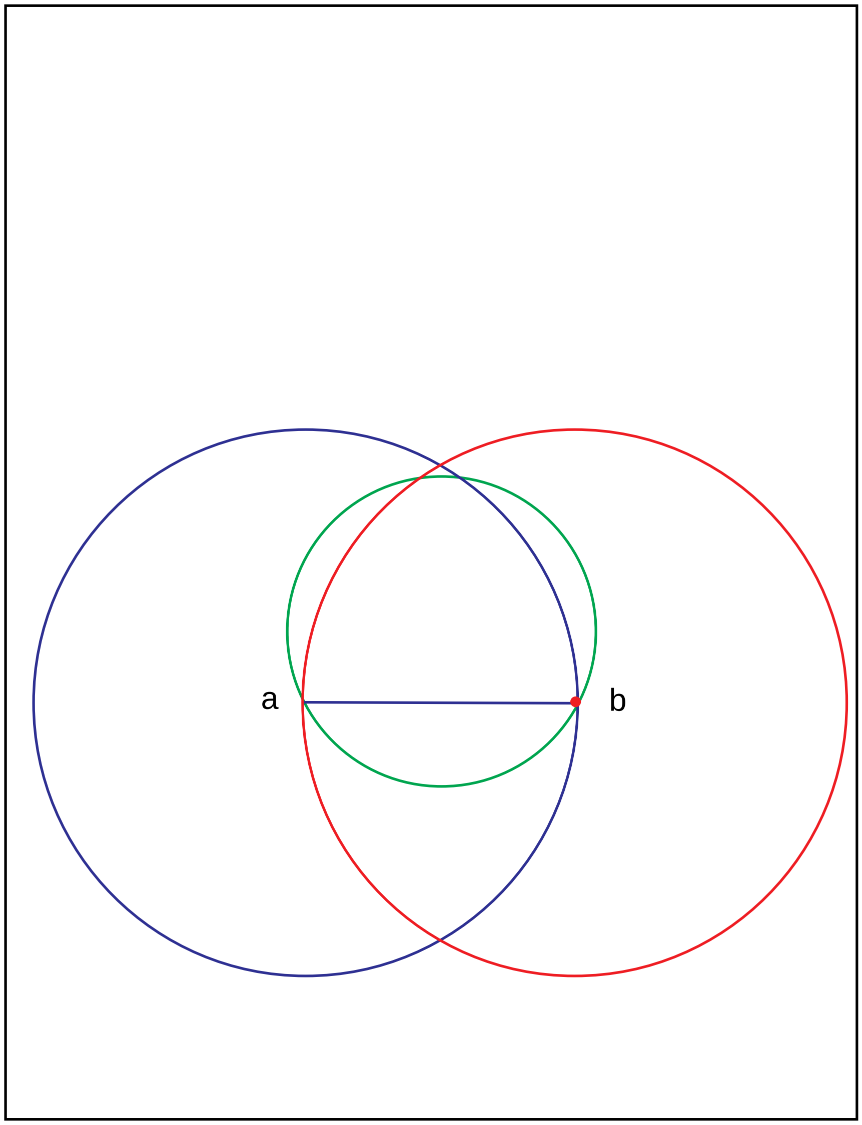


III.1:5. and let it be bisected at the point d; ([I.10])

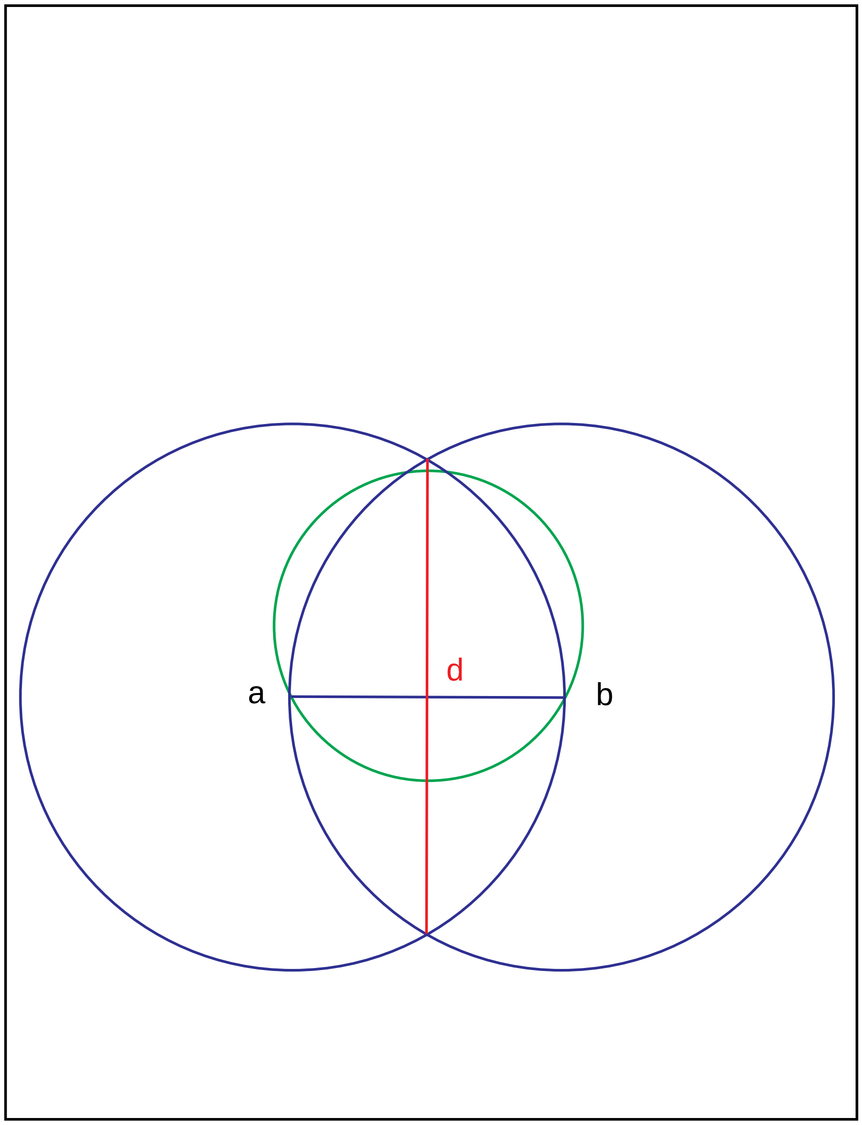
GOSUB I.10 (C#5B)
Swing ab around a.



Swing ba around b.



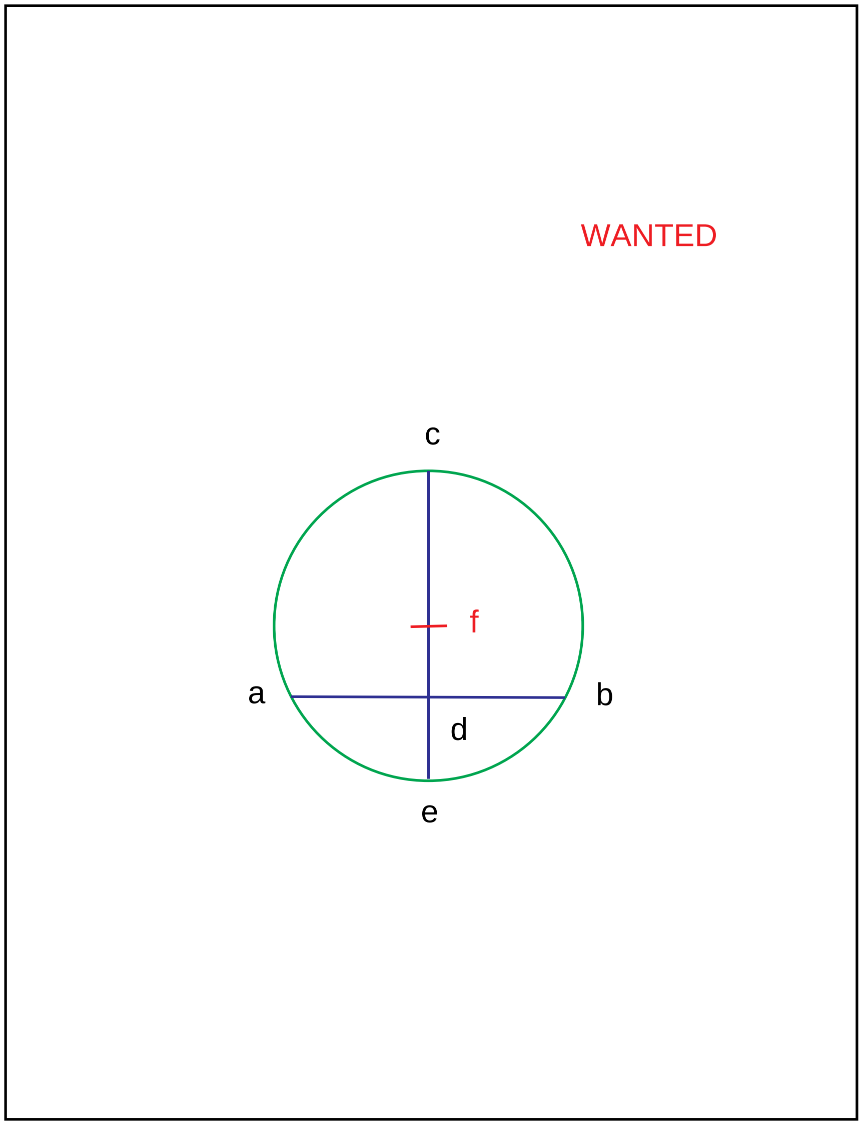
Connect the crossing points.
Mark the point d which is the
bisector of ab.



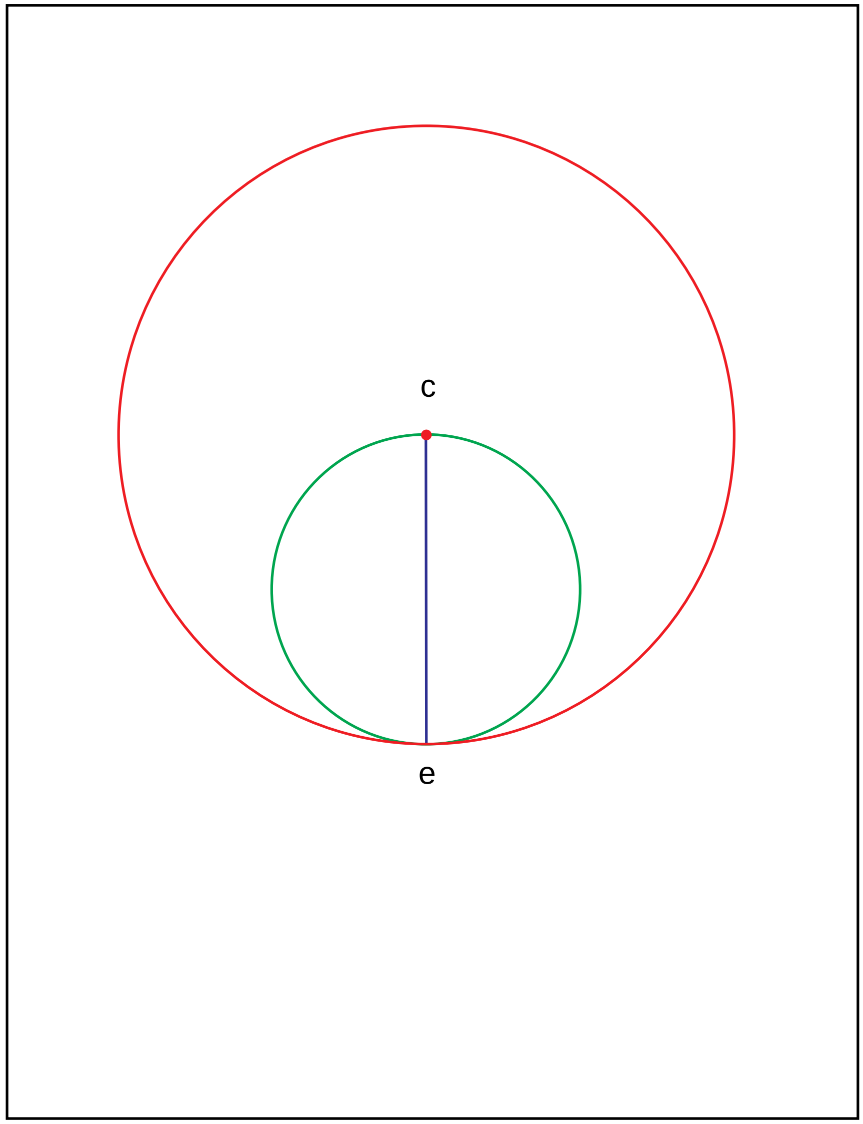
Cleanup, (preserve the new line
in anticipation).
RETURN to III.1:5.

III.1:7. from d let dc be drawn through at right angles to ab and let it be drawn through to e; let ce be bisected at f; ([I.10])

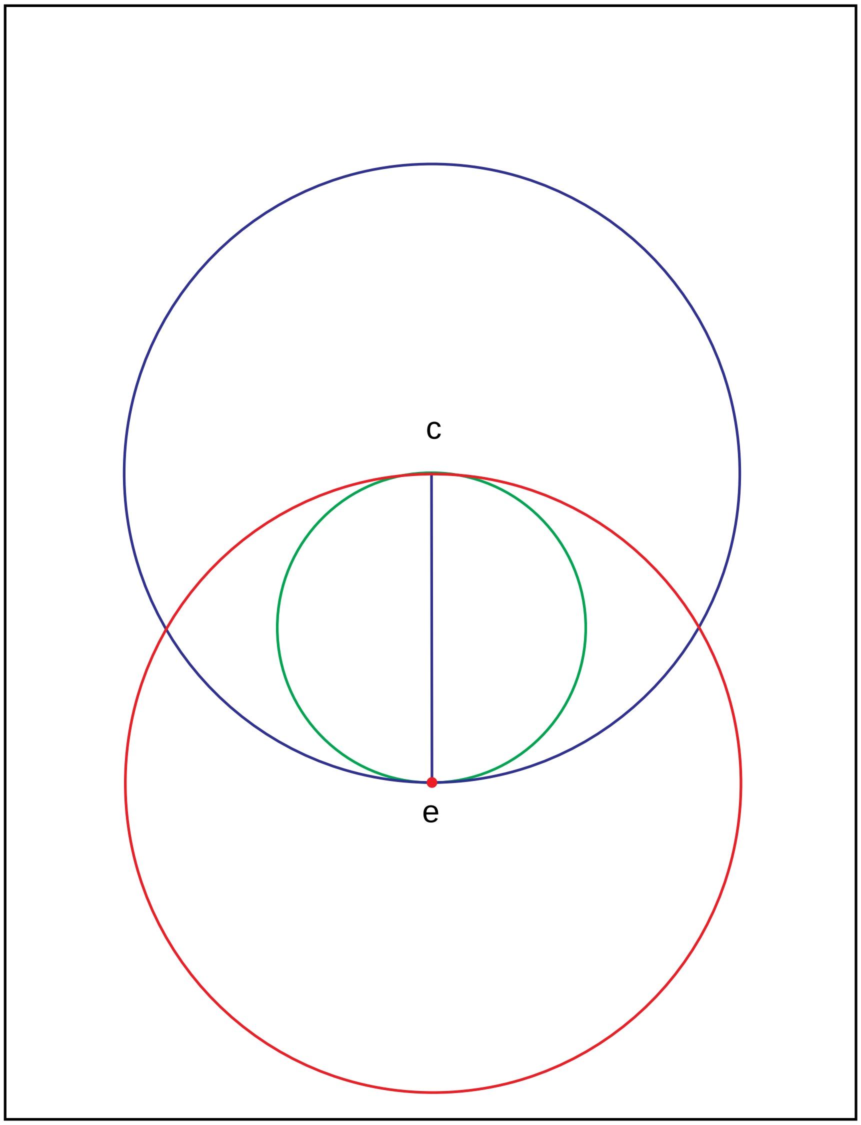
GOSUB I.10 (C#5B, again).



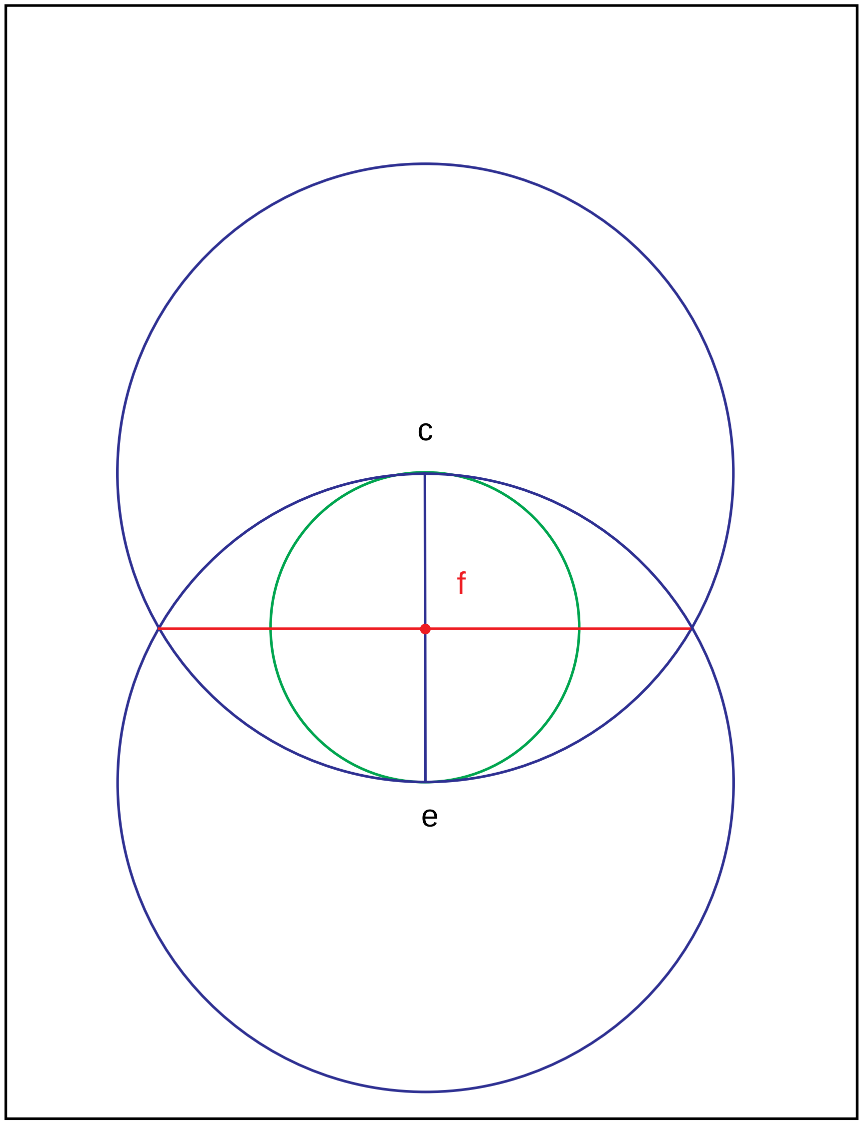
Swing ce around c.



Swing ec around e.



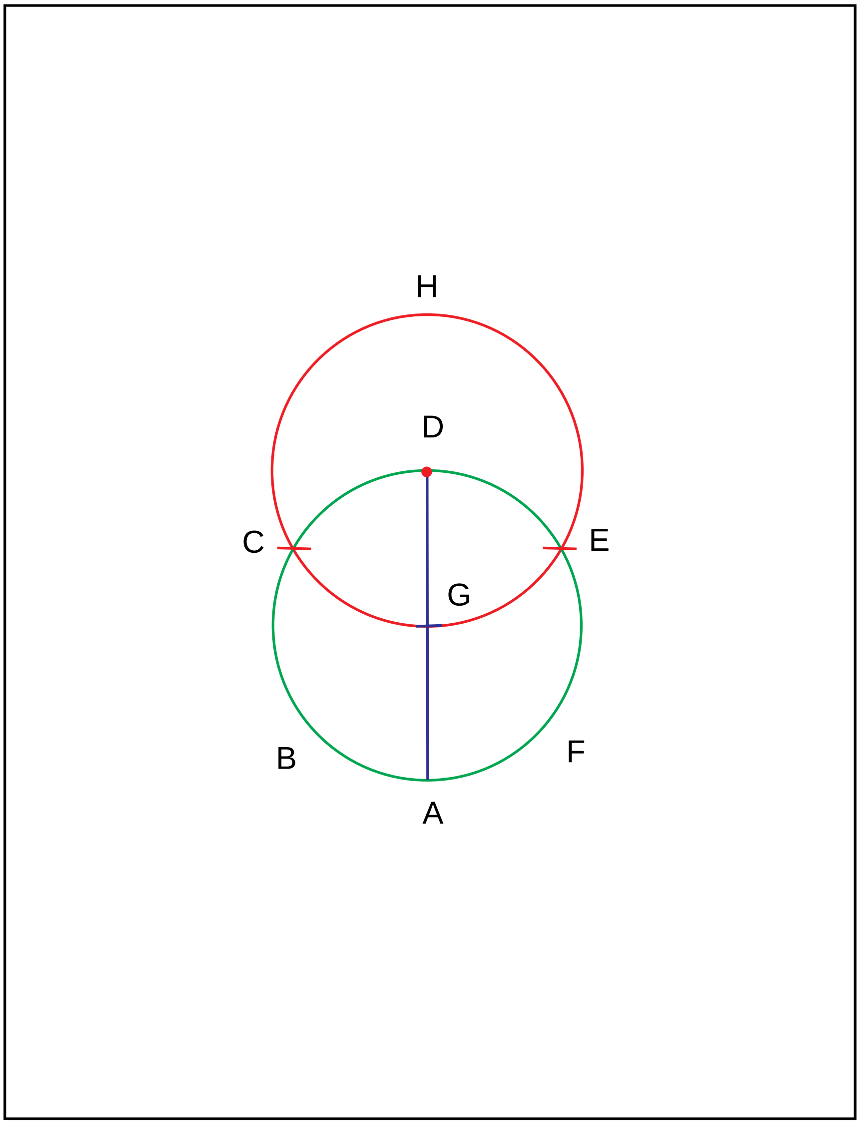
Connect the crossing points.
Mark the point f.



Cleanup. Preserve ce.
RETURN to III.1:7.
RETURN to IV.15:6.
Relabel.

A and D now label the endpoints of the diameter.

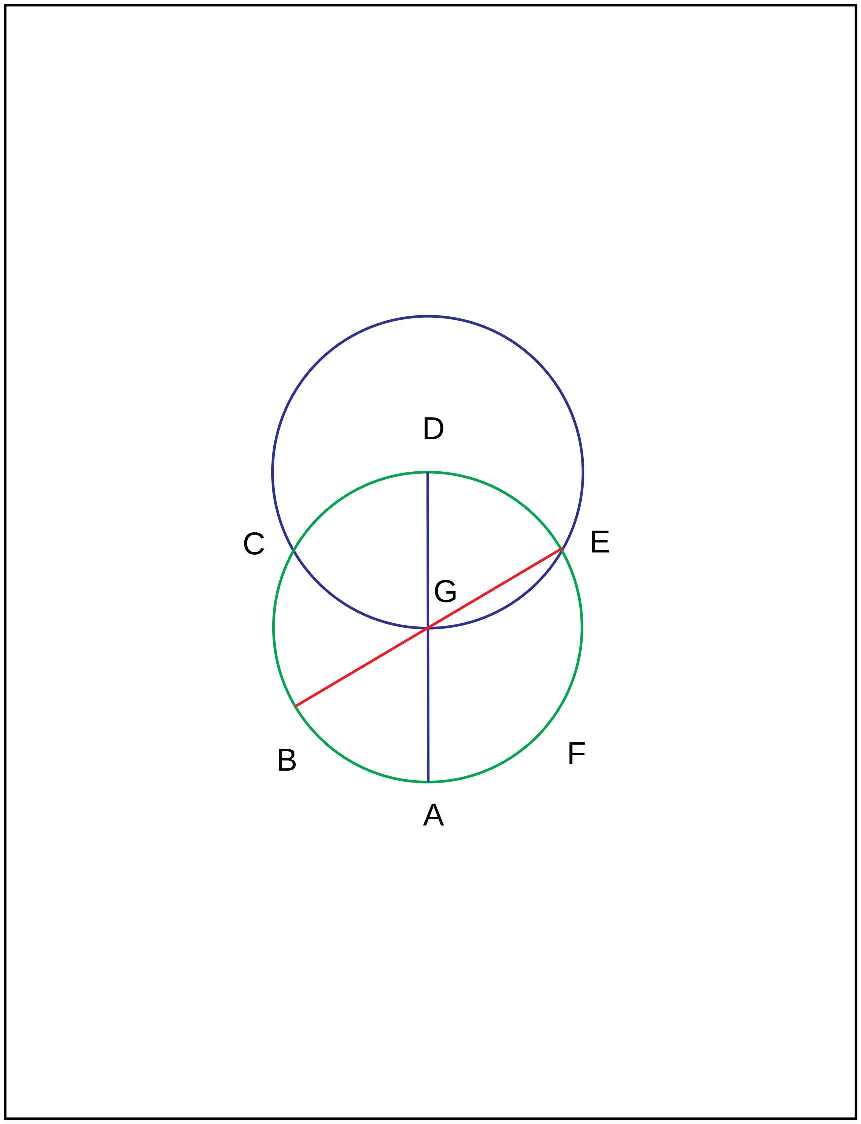
V.15:8. and with centre D and distance DG let the circle EGCH be described;



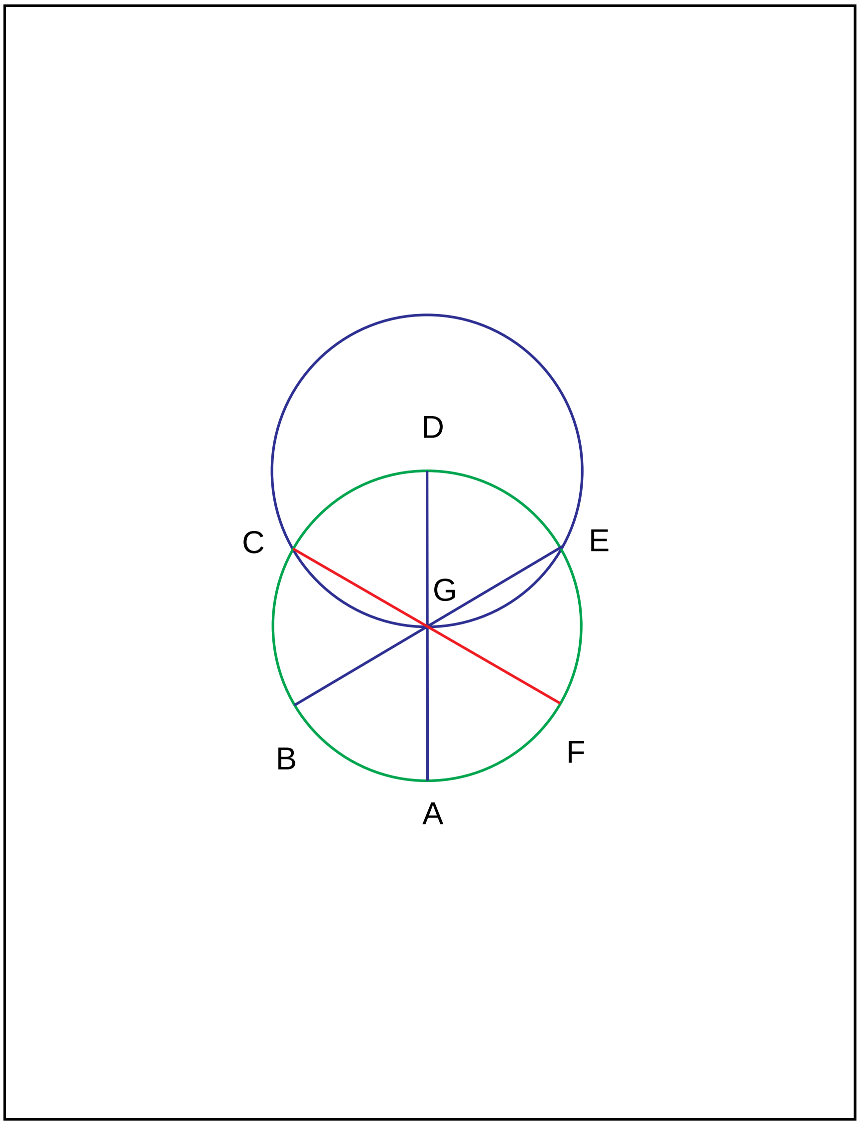
C and E now label the crossing points of the two circles.

IV.15:11. let EG, CG be joined and carried through to the points B, F,

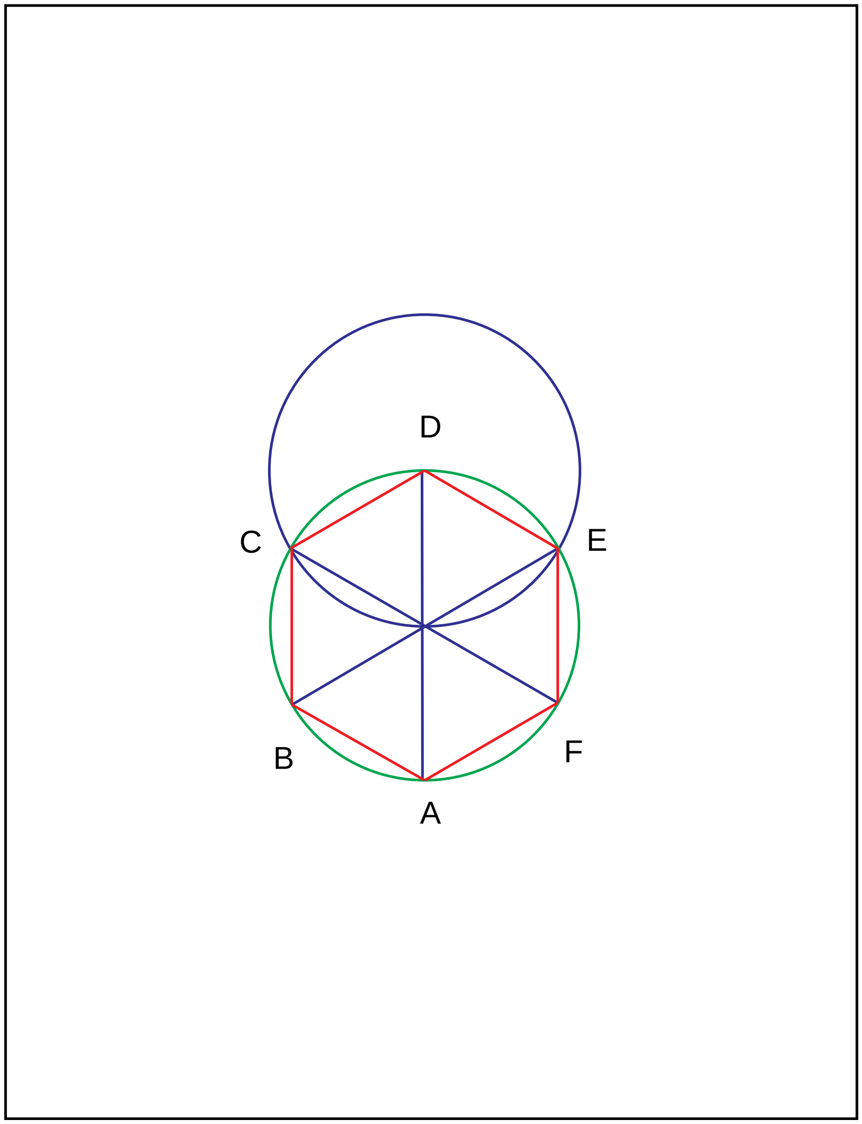
(First, EG to EB)



(Next, CG to CF)



IV.15:13. and let AB, BC, CD, DE, EF, FA be joined.

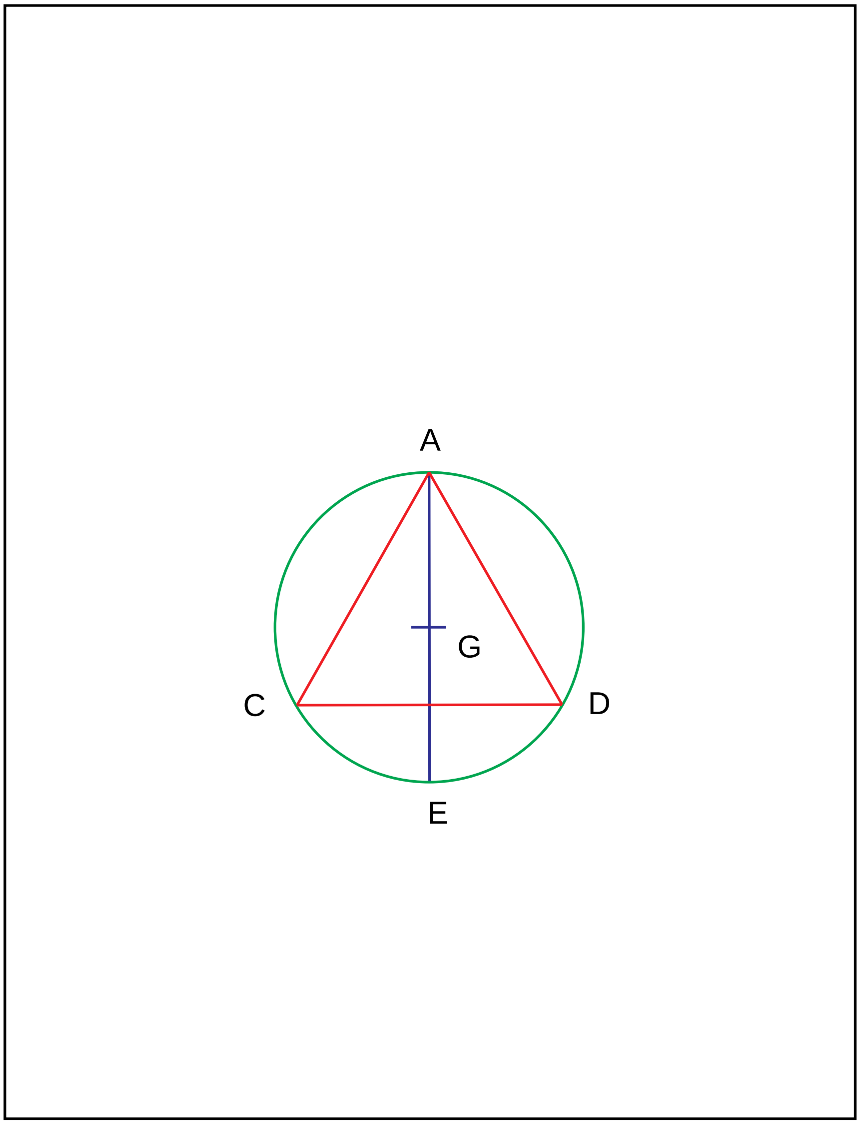


This is the hexagon, but we do not need it.

Cleanup.

Connect BD, DF, FB.

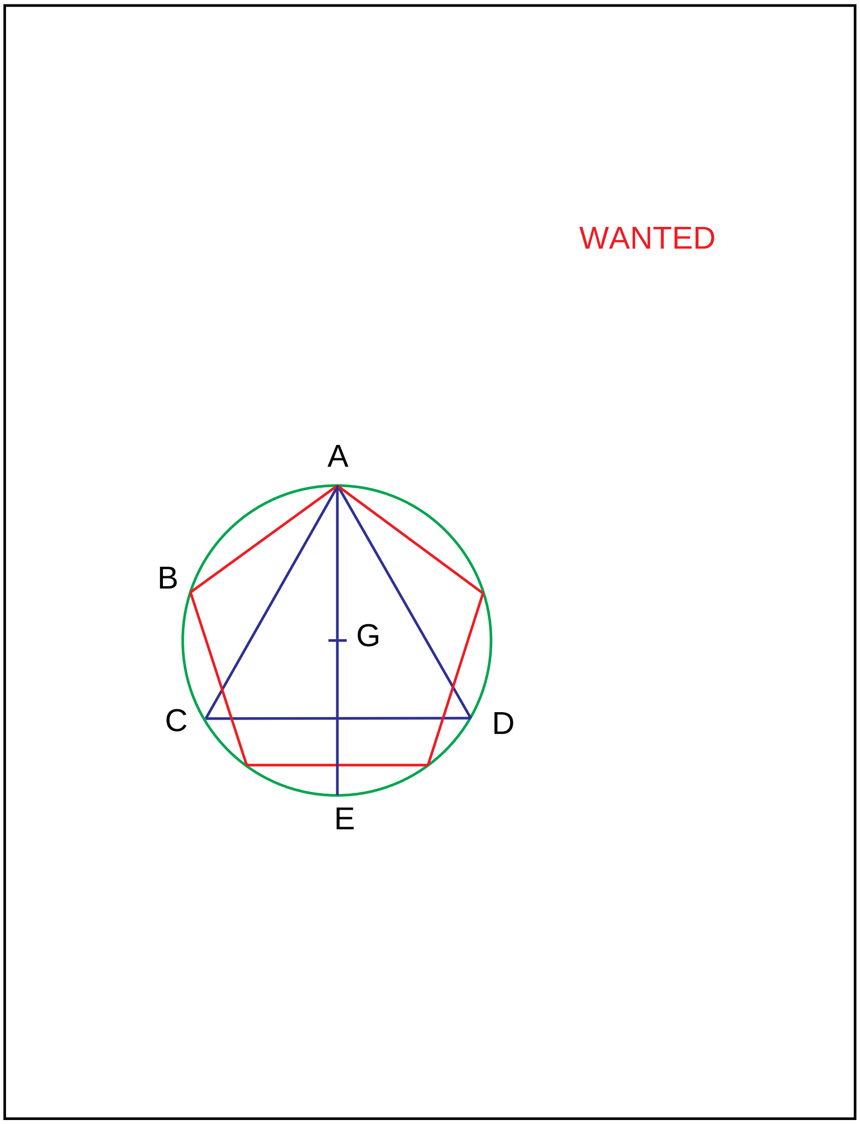
Cleanup. Preserve ad and g.
Relabel.
RETURN to IV.16:8.



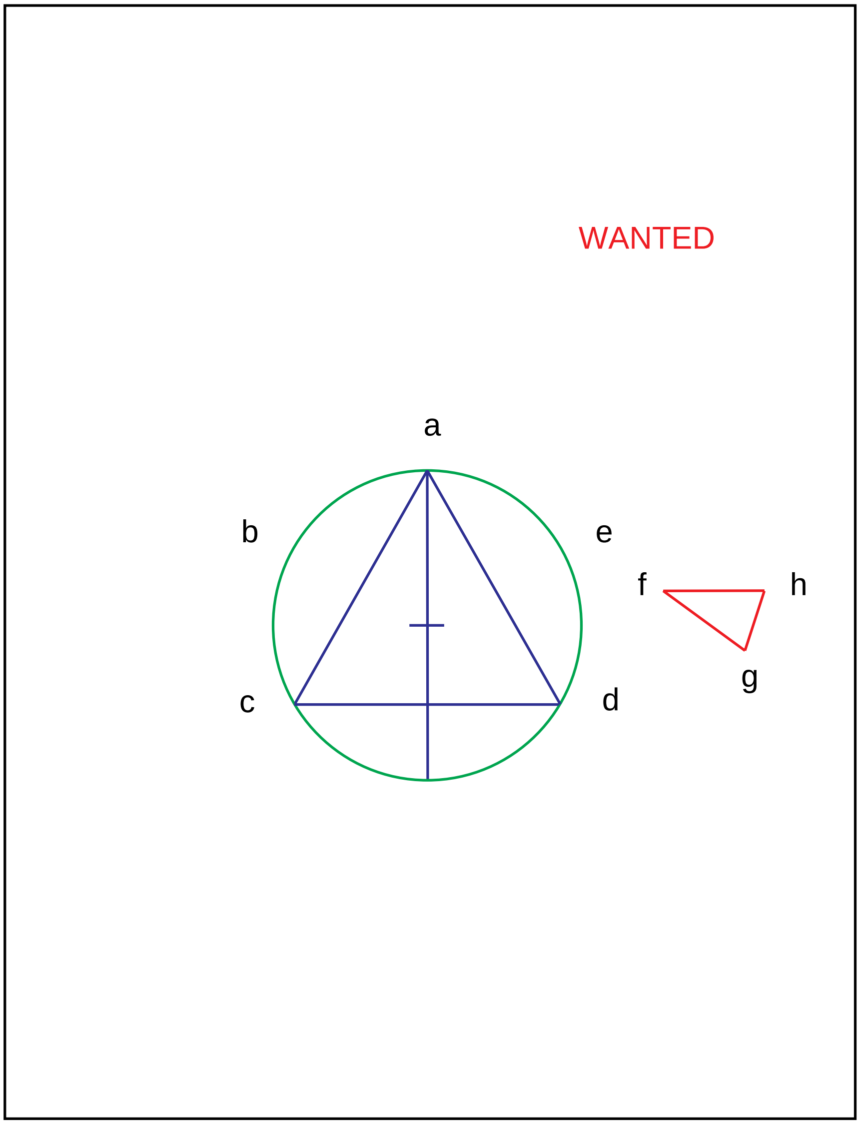
IV.16:11. and a side AB of an equilateral pentagon; ([IV.11])

GOSUB IV.11.
Relabel.

NOTE: The point A is fixed in advance. A diameter at A will be wanted early in IV.11, and a centre. For this reason we saved AE and G above. This will save 8 steps later.

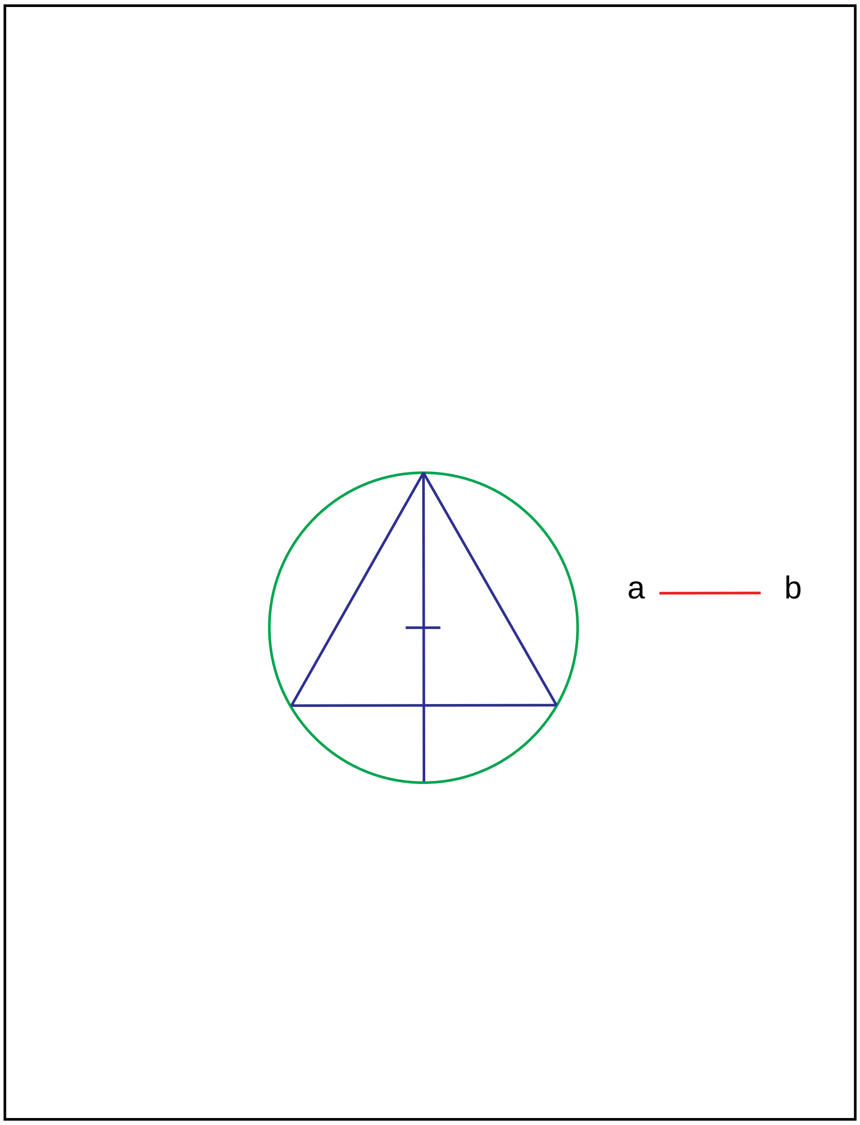


IV.11:6. Let the isoceles triangle fgh be set out having each of the angles at g, h double of the angle at f; [IV.10]

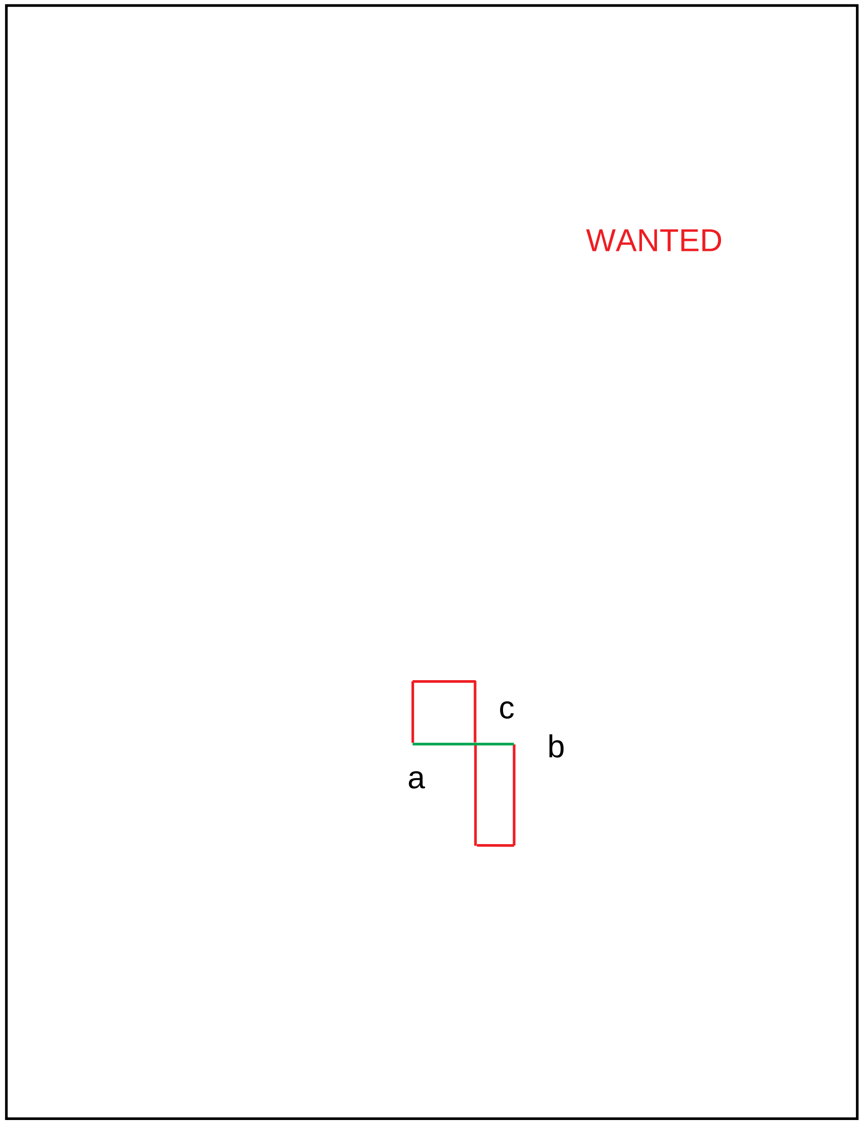


GOSUB IV.10.
Relabel.

IV.10:3. Let any straight line ab be set out,

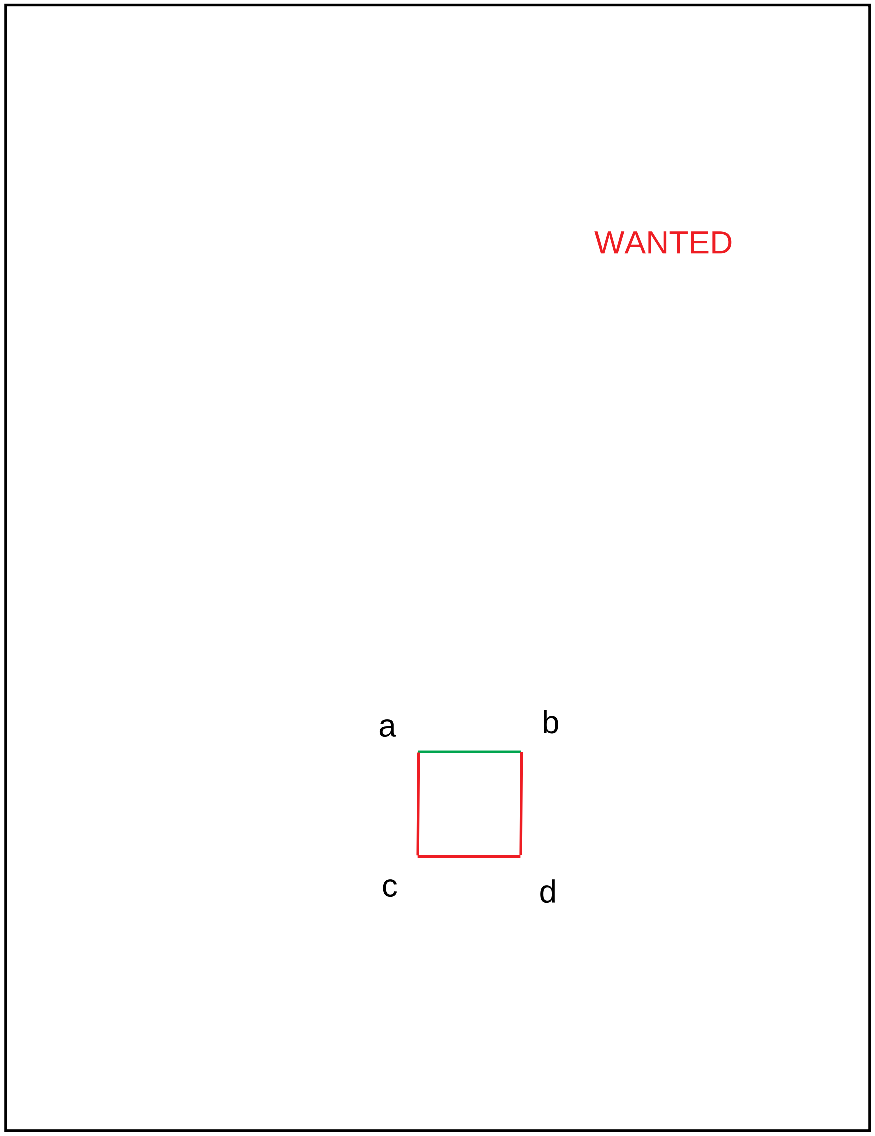


IV.10:3. and let it be cut at the point c so that the rectangle contained by ab, bc is equal to the square on ca; [II.11]

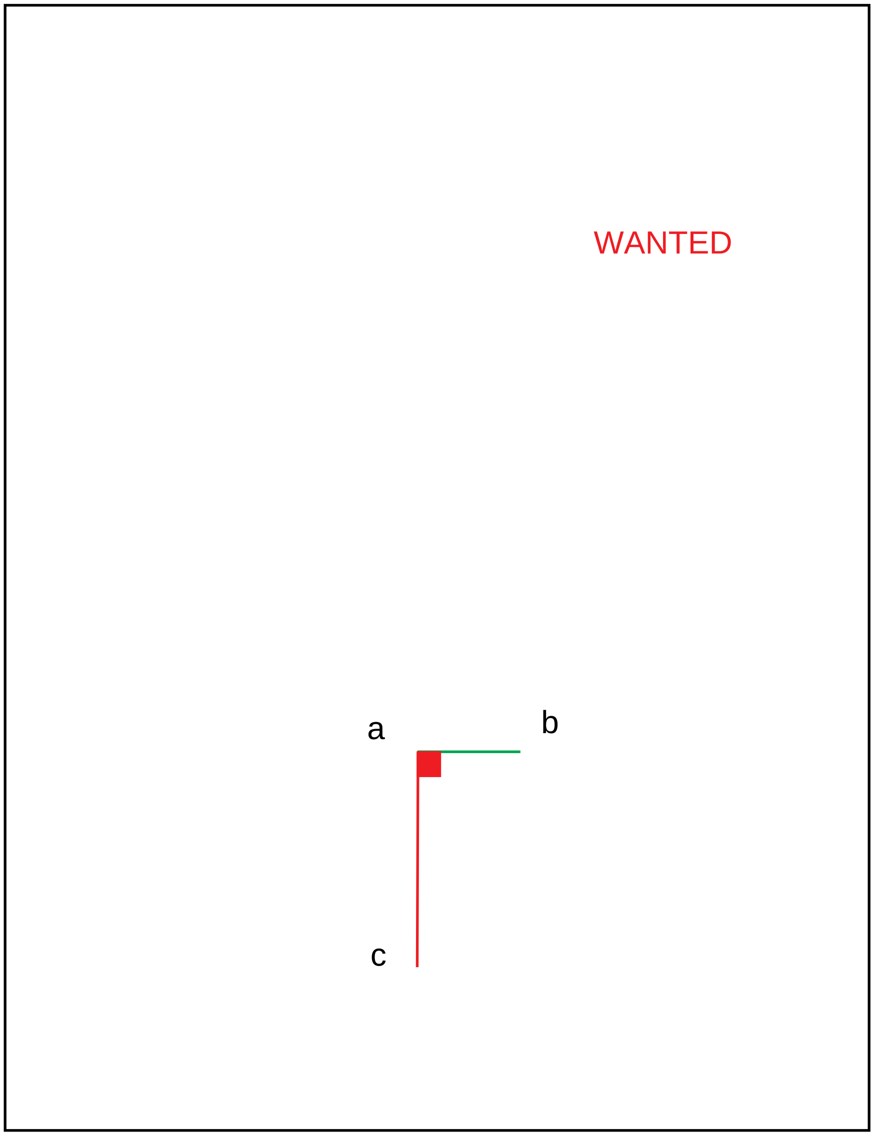


II.11:9. For let the square abcd be described on ab; [I.46].

GOSUB I.46.



I.46:5. Let ac be drawn at right angles to the straight line ab from the point a on it [I.11],



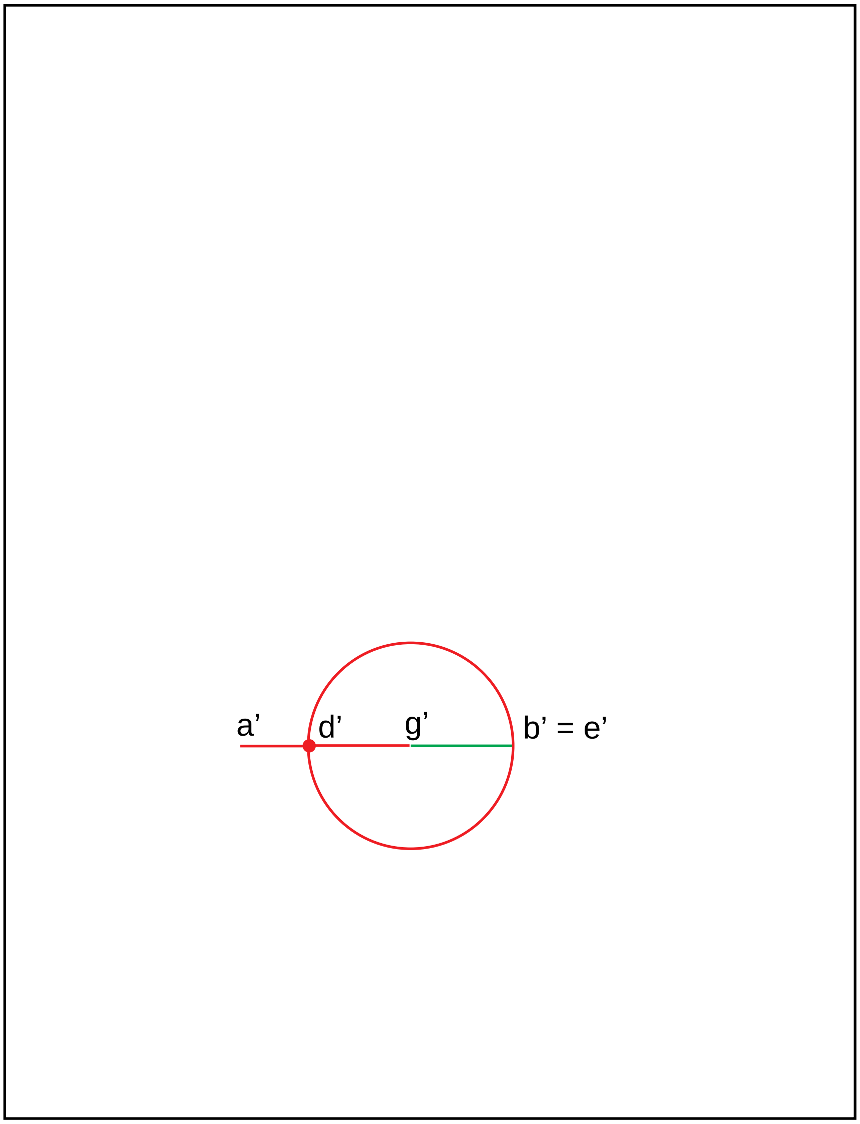
GOSUB I.11.
Extend the line.
Relabel.

GOSUB II.11. Extend the line.
Relabel.

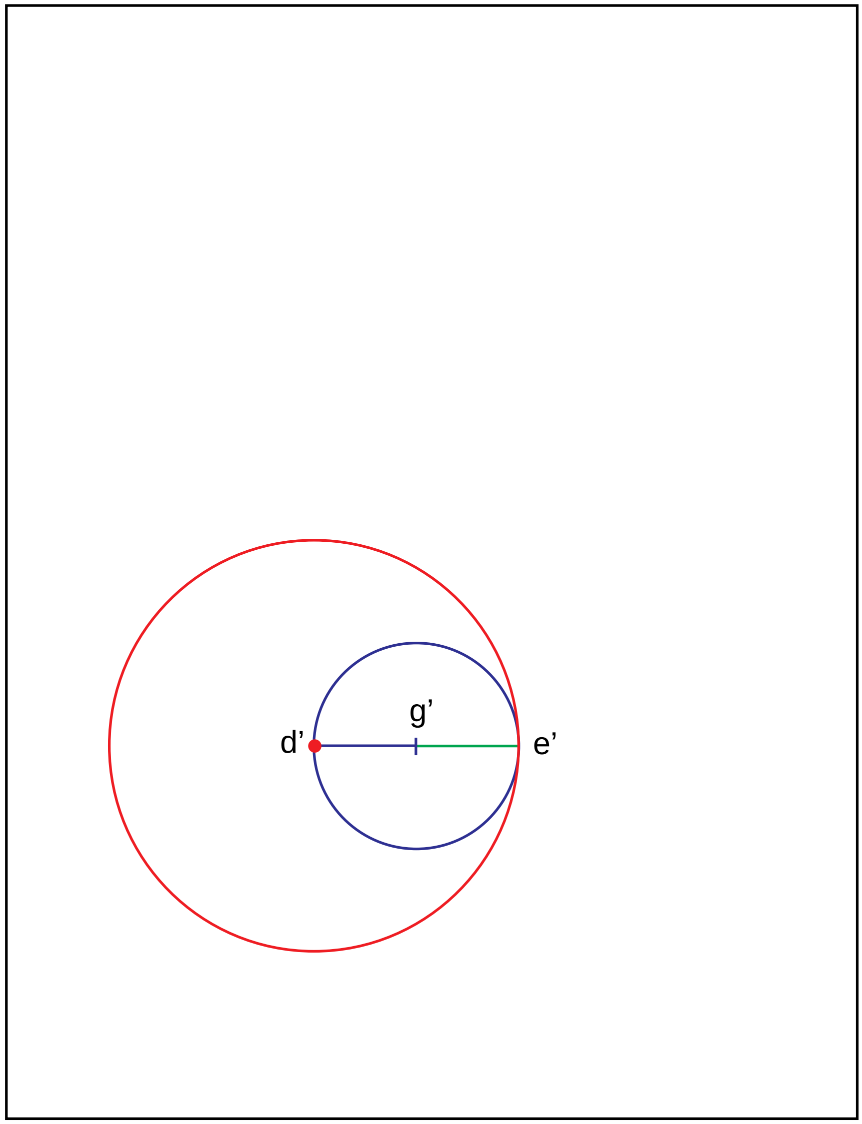
II.11:8. Let a point d’ be taken at random on a’g’;
let g’e’ be made equal to g’d’;
[I.3]

NOTE. Rather than use dividers in place of I.3 as usual (cf. I.3) it will be useful here to use the compass, and draw a circle as shown.

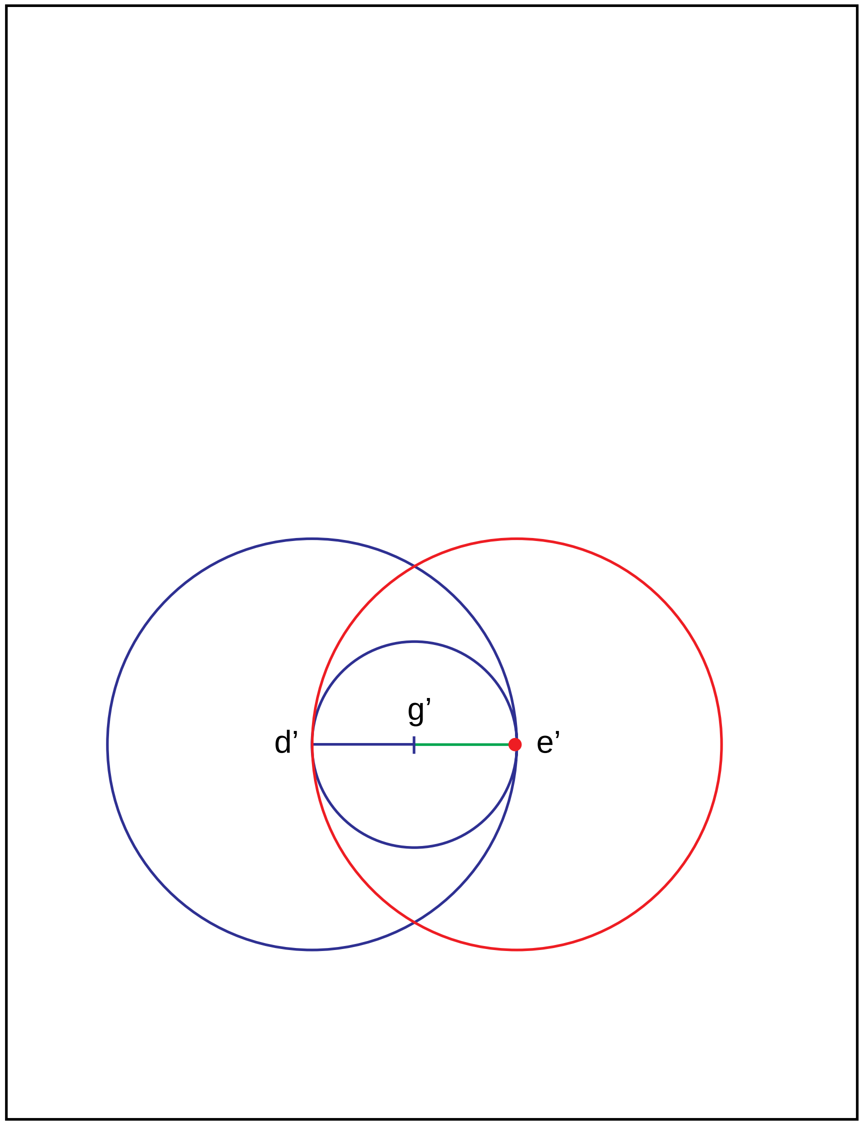
Choose e’ = b’.
With centre g’ and distance g’e’ let the circle be described. Let d’ be the point through which this circle crosses a’b’.



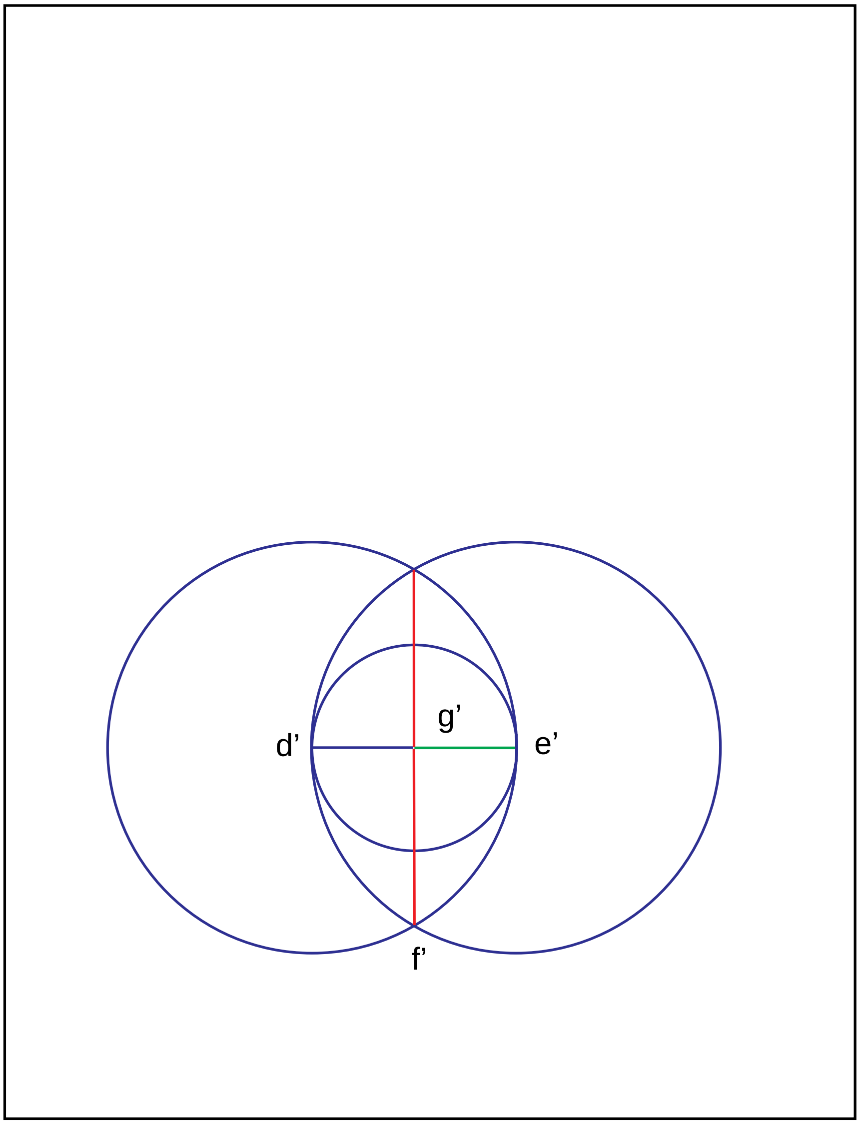
With centre d’ and distance d’e’ let the circle be described; [Post. 3]



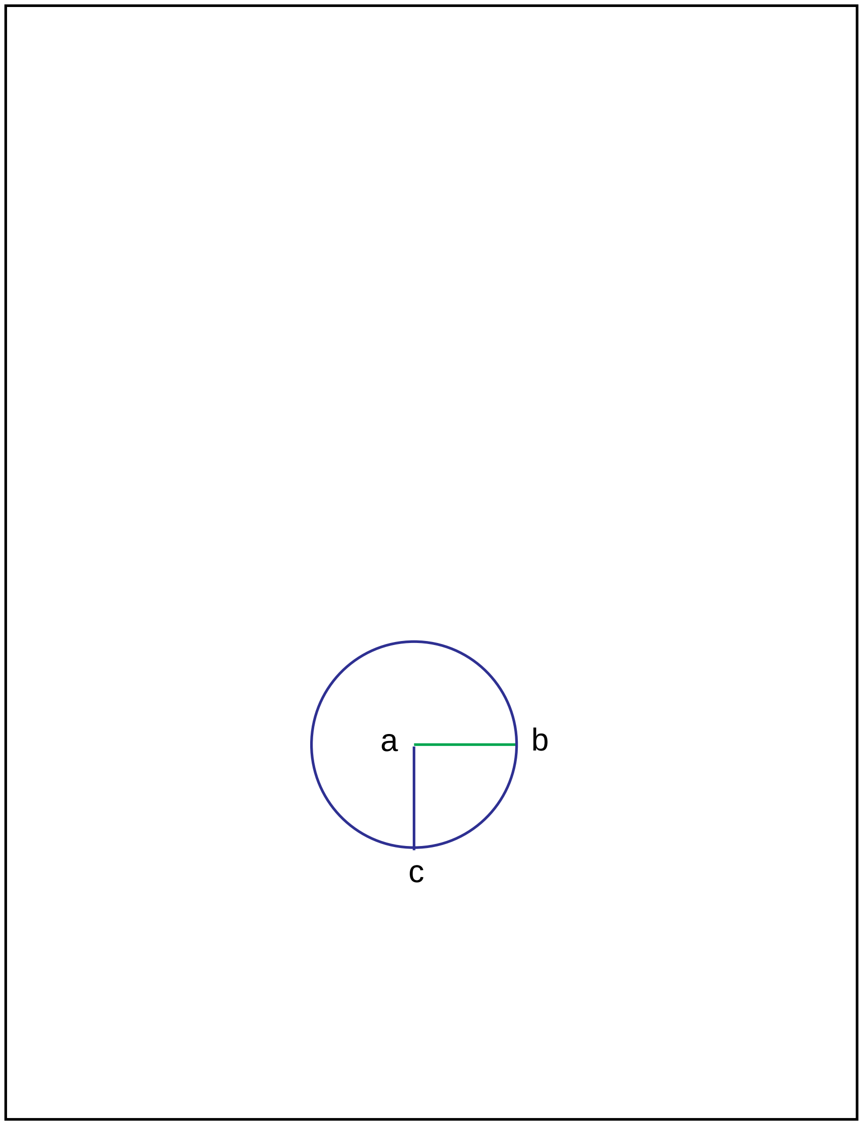
again, with centre e’ and distance e’d’ let the circle be described; [Post. 3]



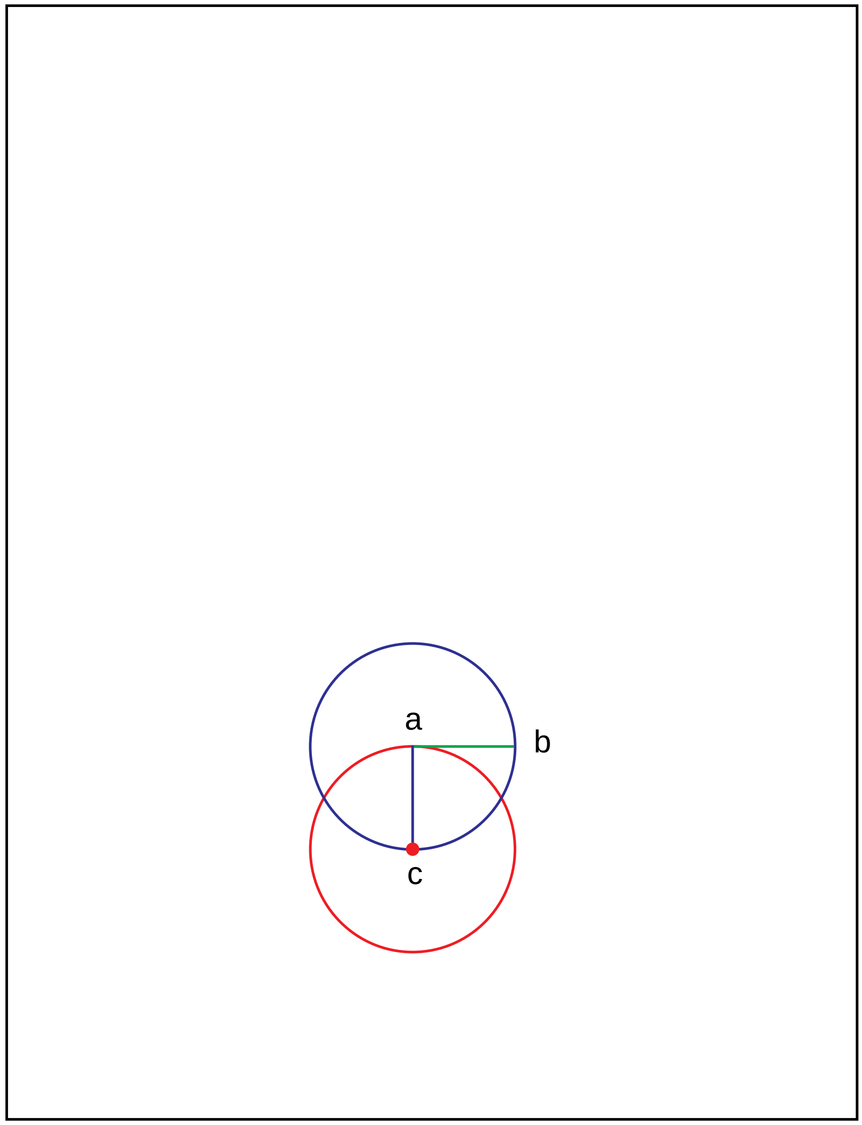
nd let f’g’ be joined; I say that the straight line f’g’ has been drawn at right angles to the given straight line a’b’ from g’ the given point on it.

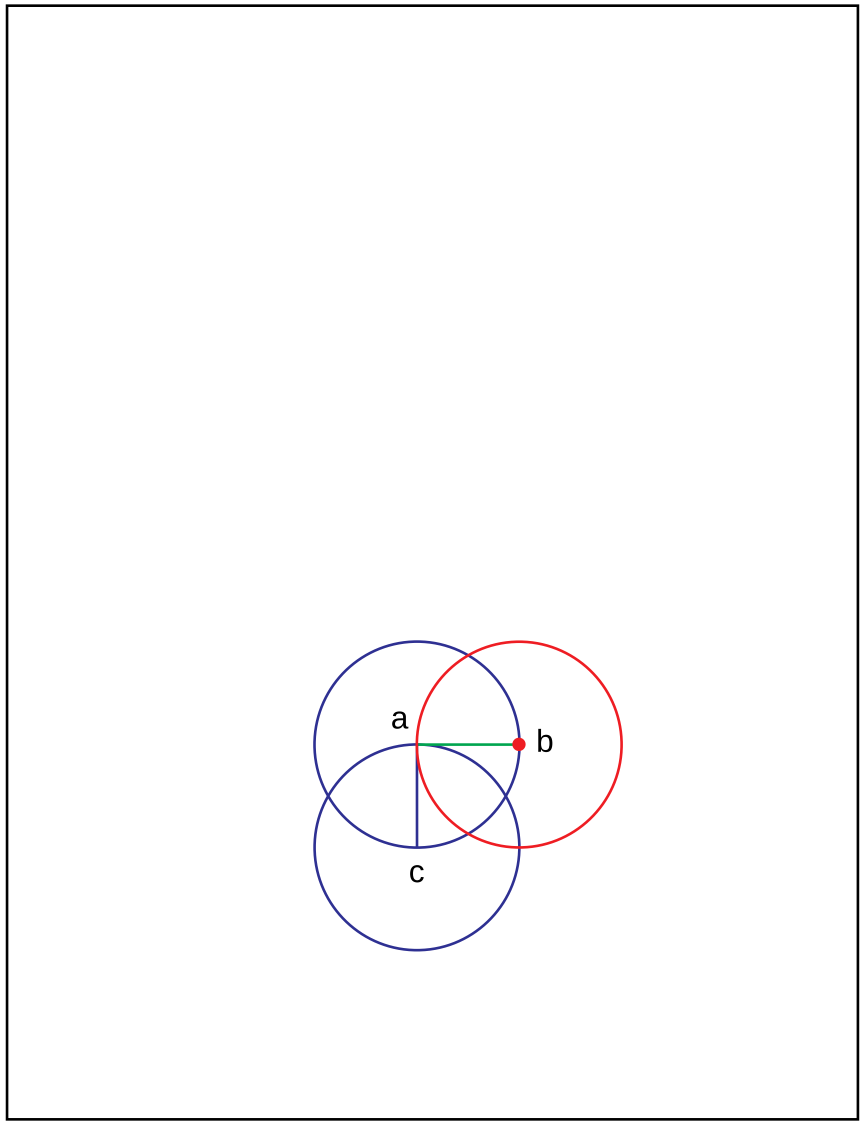


and let ac be made equal to ab;

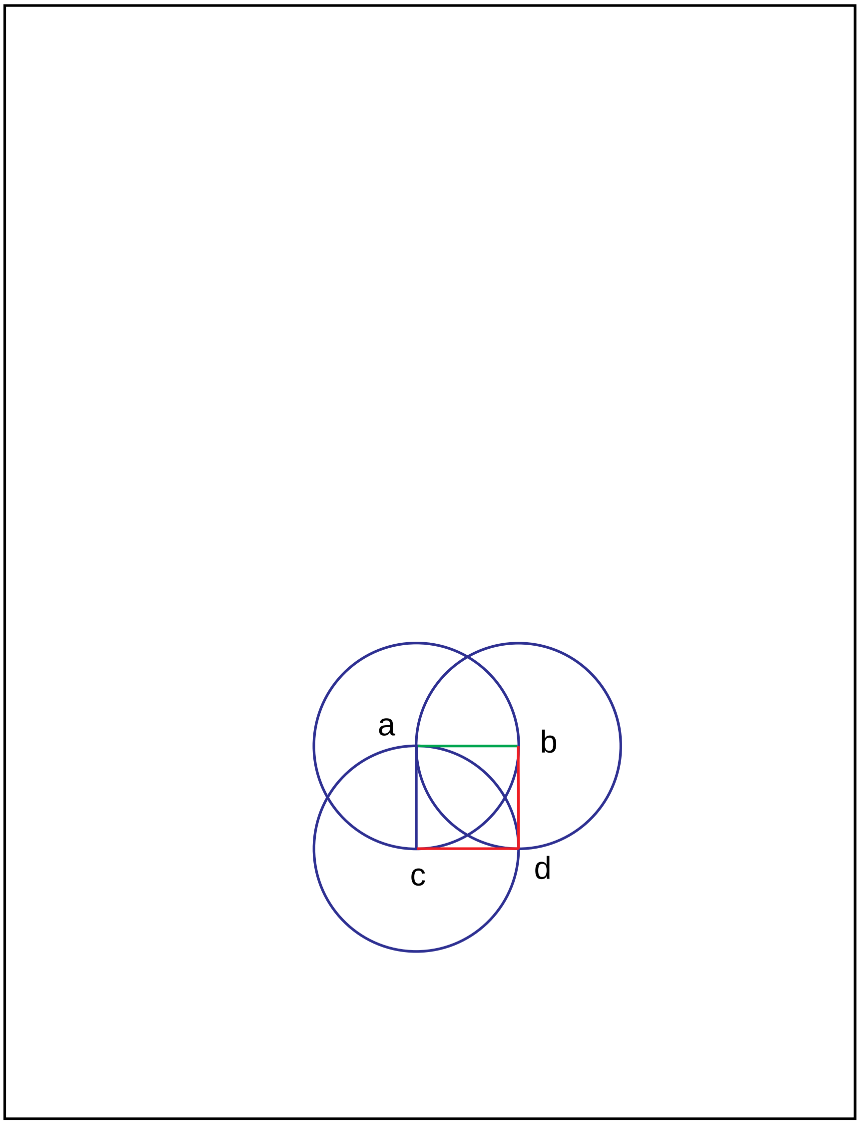


Swing da around d.


a
b
c


a
b
c

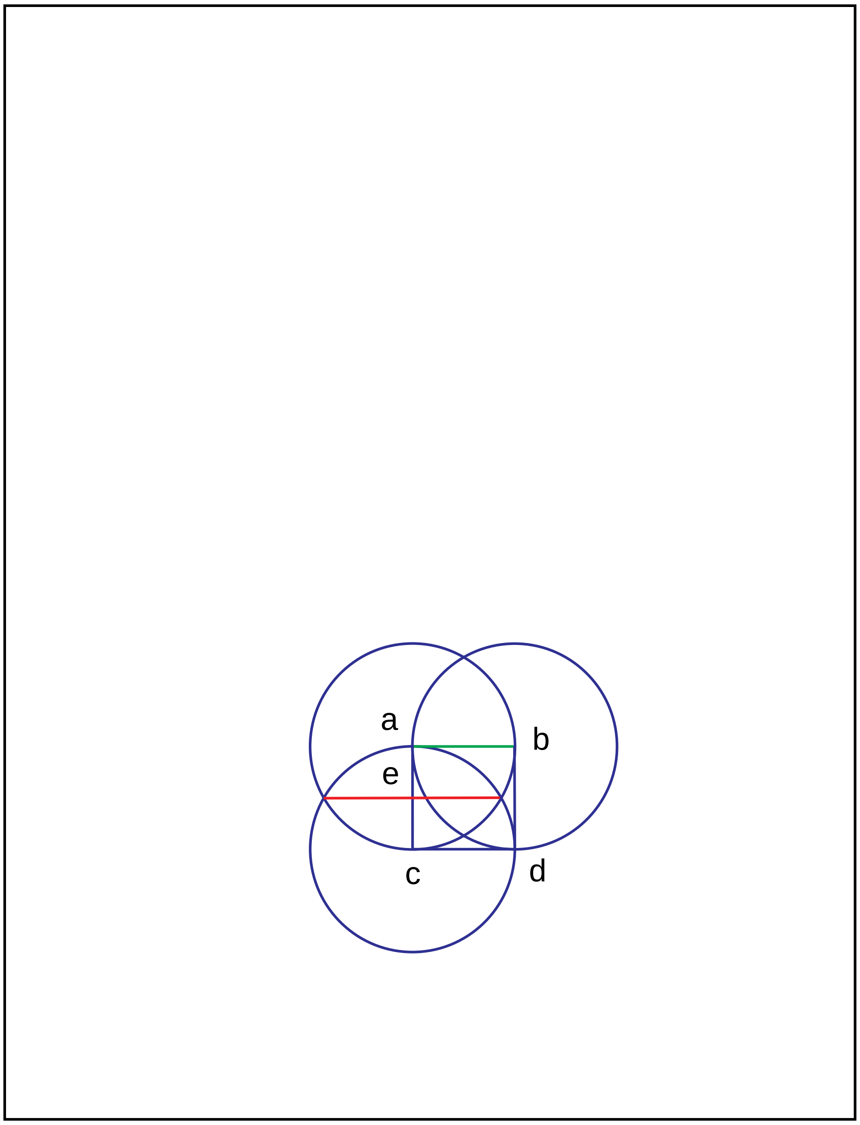
Connect the crossing point e to the centers.



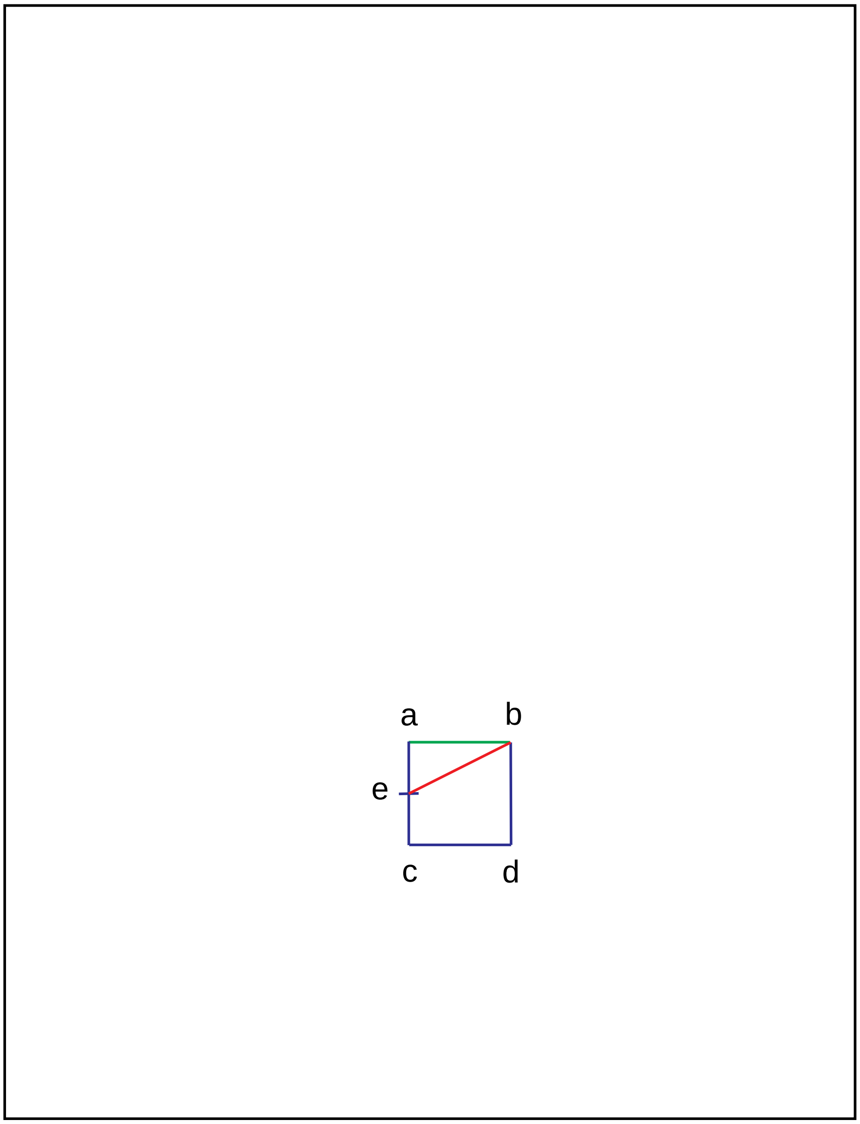
I.1:7. With centre a and distance ab let the circle be described;
(Revive step 6)

I.1:10. again, with centre b and distance ba let the circle be described;
(Revive step 9)

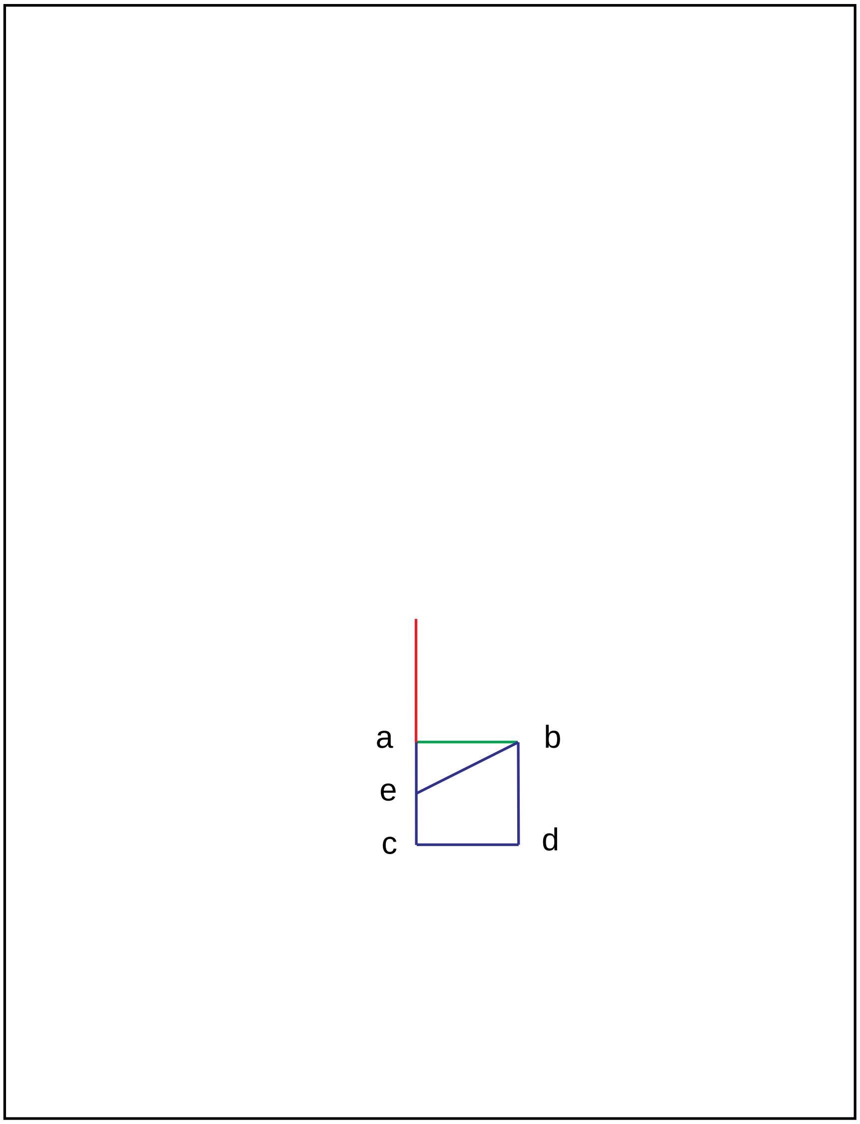
Connect the crossing points.
Mark the point where this line crosses ab.
Cleanup and RETURN to II.11 at line 11.



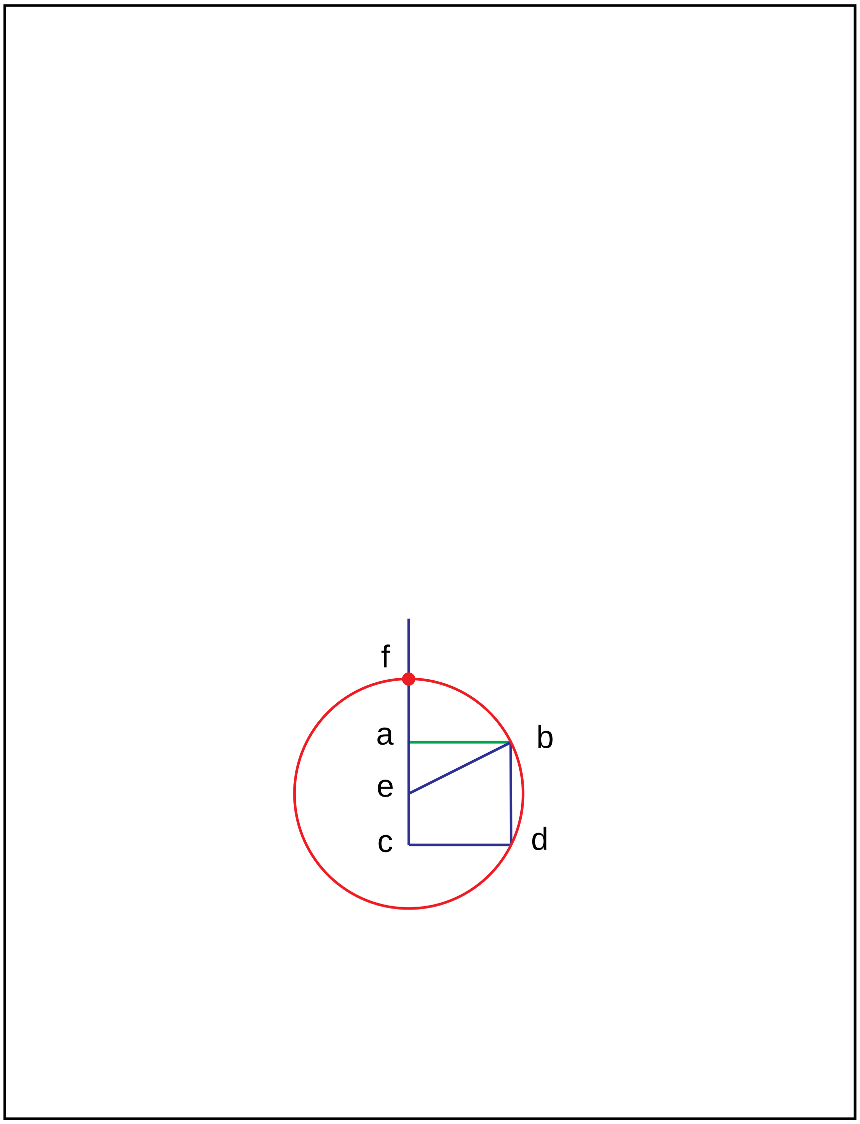
II.11:11. and let be be joined;



II.11:12. let ca be drawn through to f,
(this point is not actually located.)



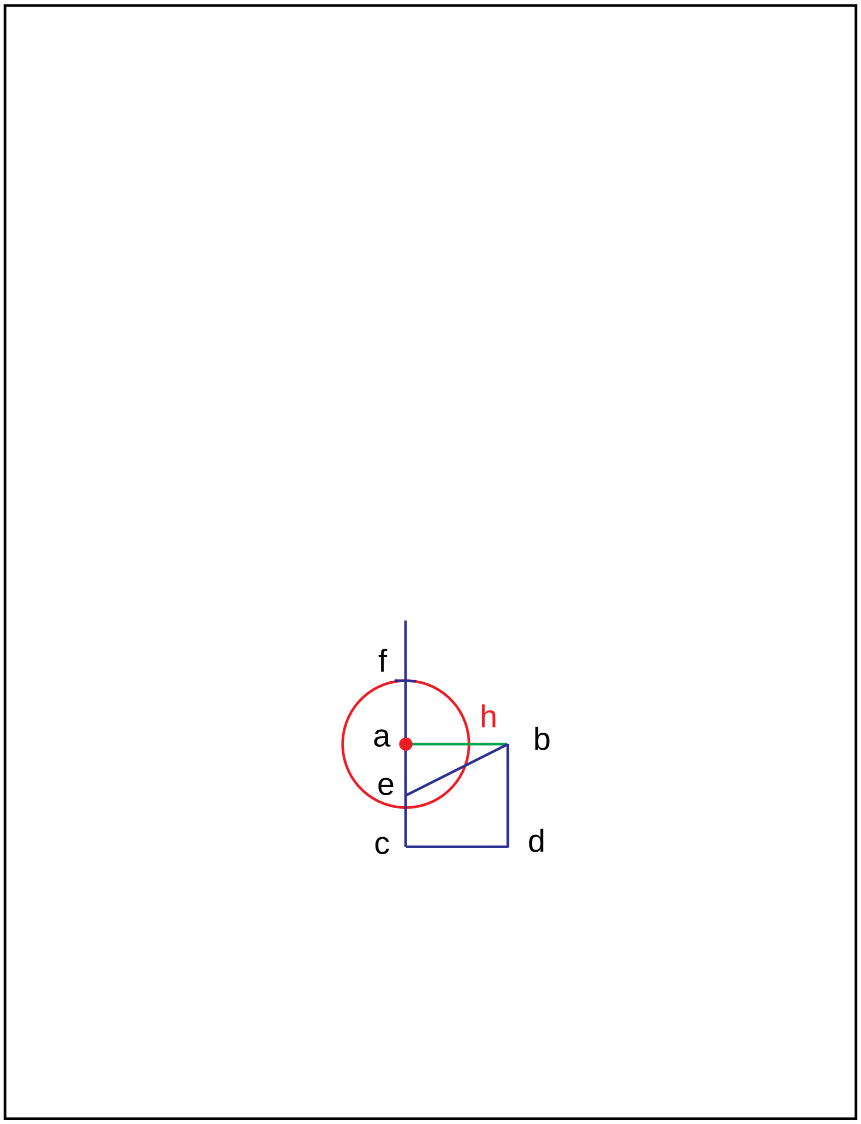
And let ef be made equal to be;
([I.3], or dividers)



Swing from the corner, a. Locate the point h on ab so ah is equal to af. (or, dividers.)

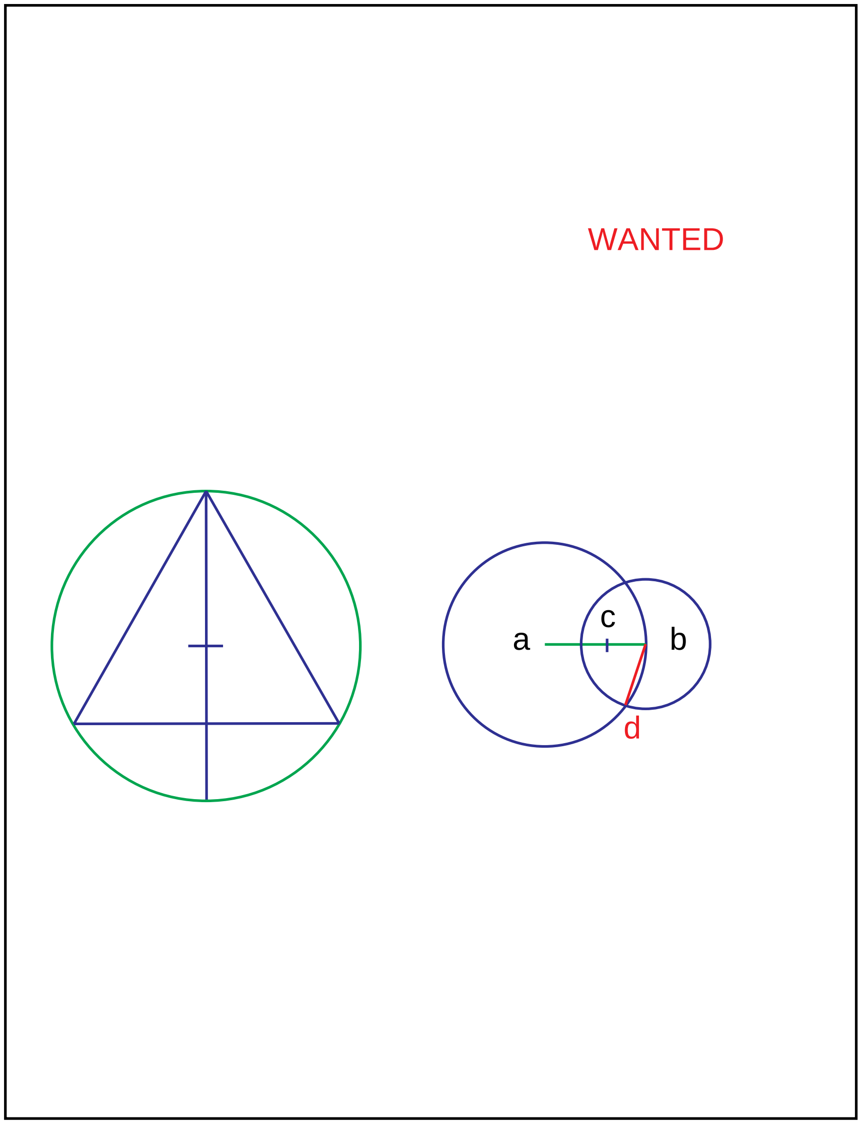
NOTE: With this point we have finished with the first instruction: To cut a given straight line. This is the golden section in 14 steps.

Cleanup.
RETURN from II.11 to IV.10:3.
Relabel



IV.10:7. With centre a and distance ab let the circle bde be described, (Restore from step 14.)

IV.10:9. and let there be fitted in the circle bde the straight line bd equal to the straight line ac... [IV.1]

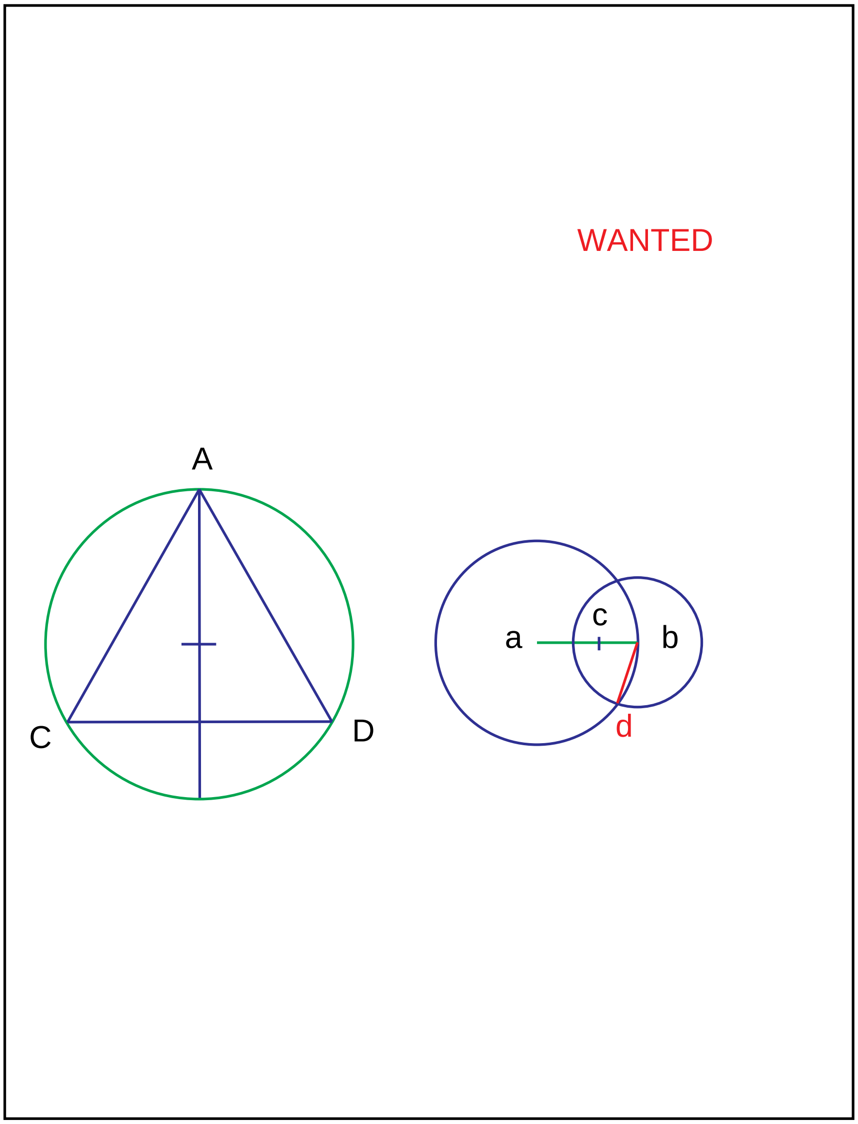


IV.10:7.

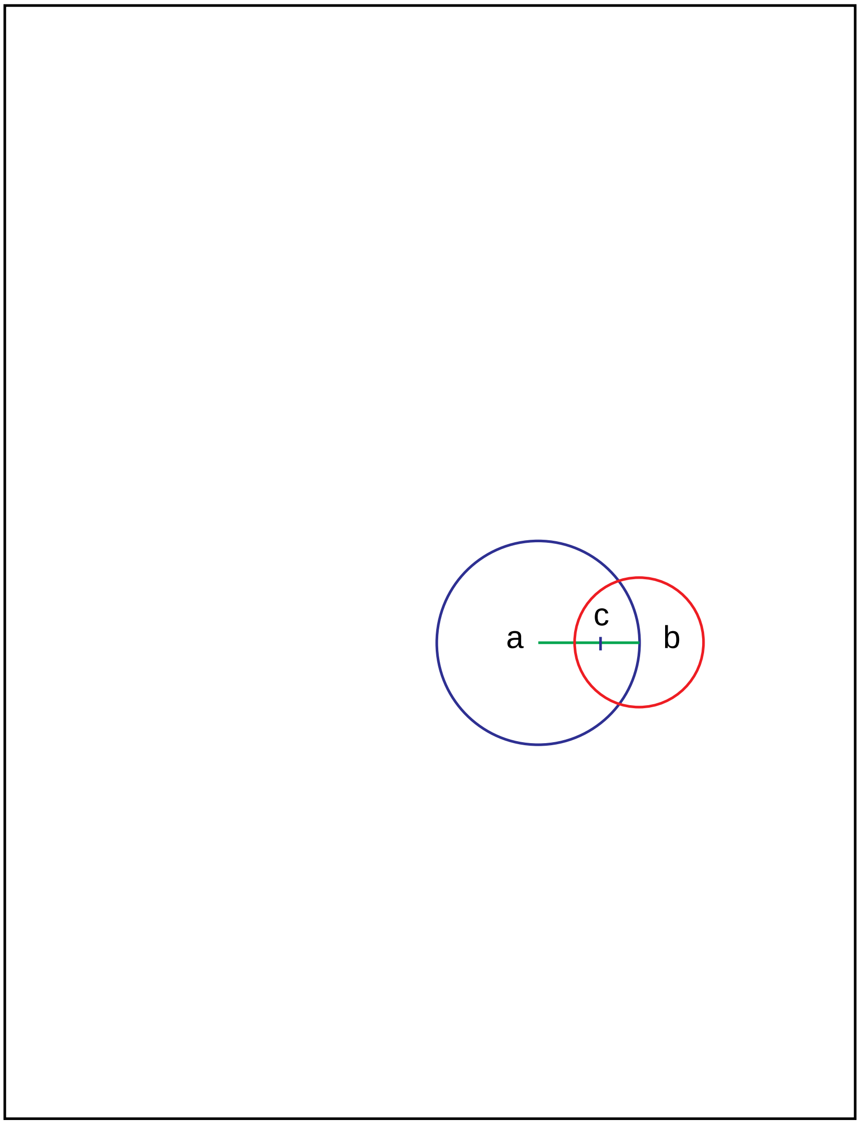
IV.10:9 = [IV.1] (C#23)

GOSUB C#23

Note. ab was a length chosen at random. We are to construct a magic triangle on it, as a first stage in constructing the inscribed pentagon. We have just finished the first task: division of ab into extreme and mean ratios.

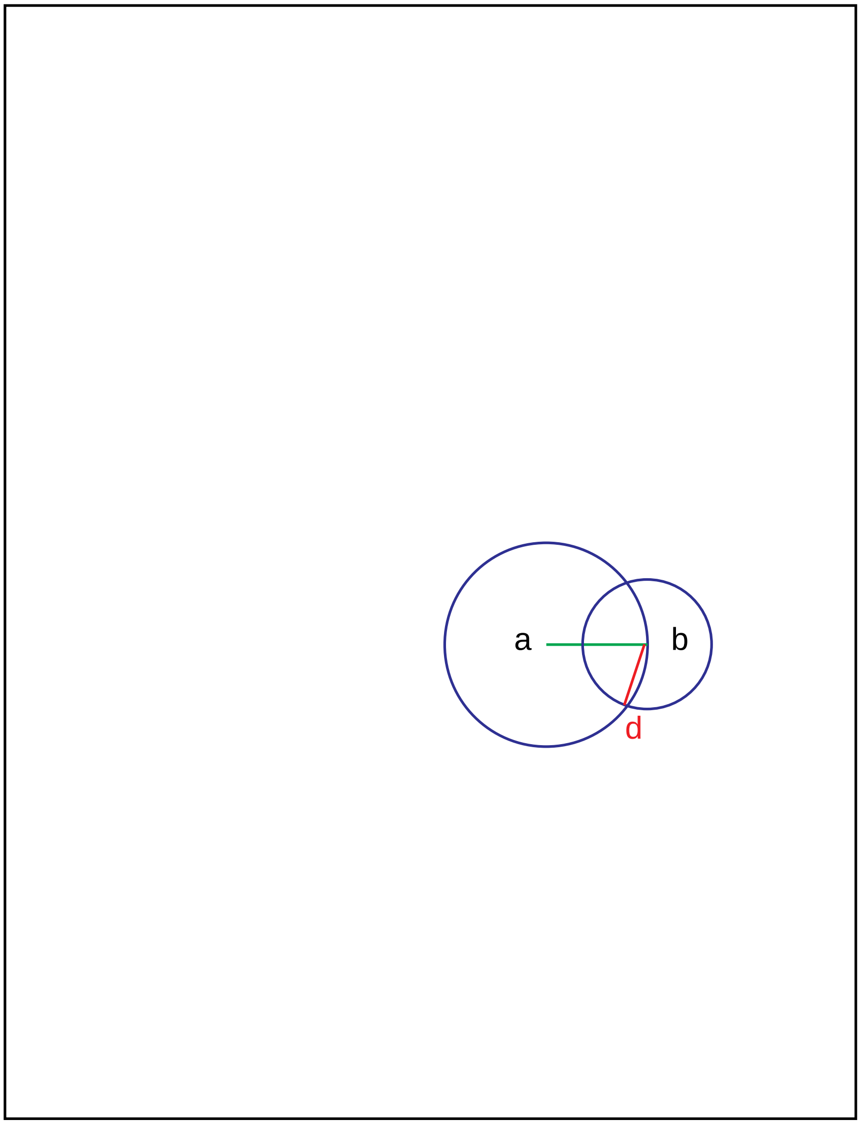


IV.1:17 (paraphrase). With centre b and distance ac let the circle be desciribed; (we set the compass and move it.)

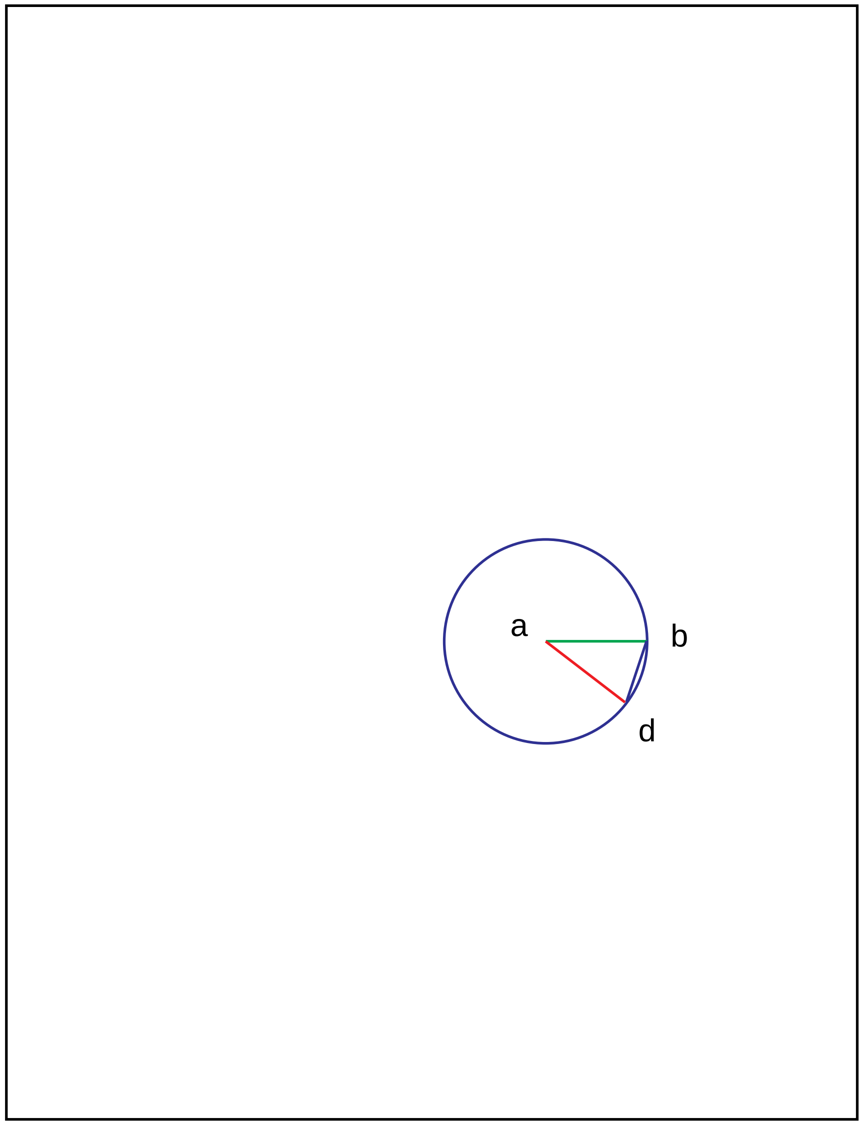


IV.1:19. let bd be joined.

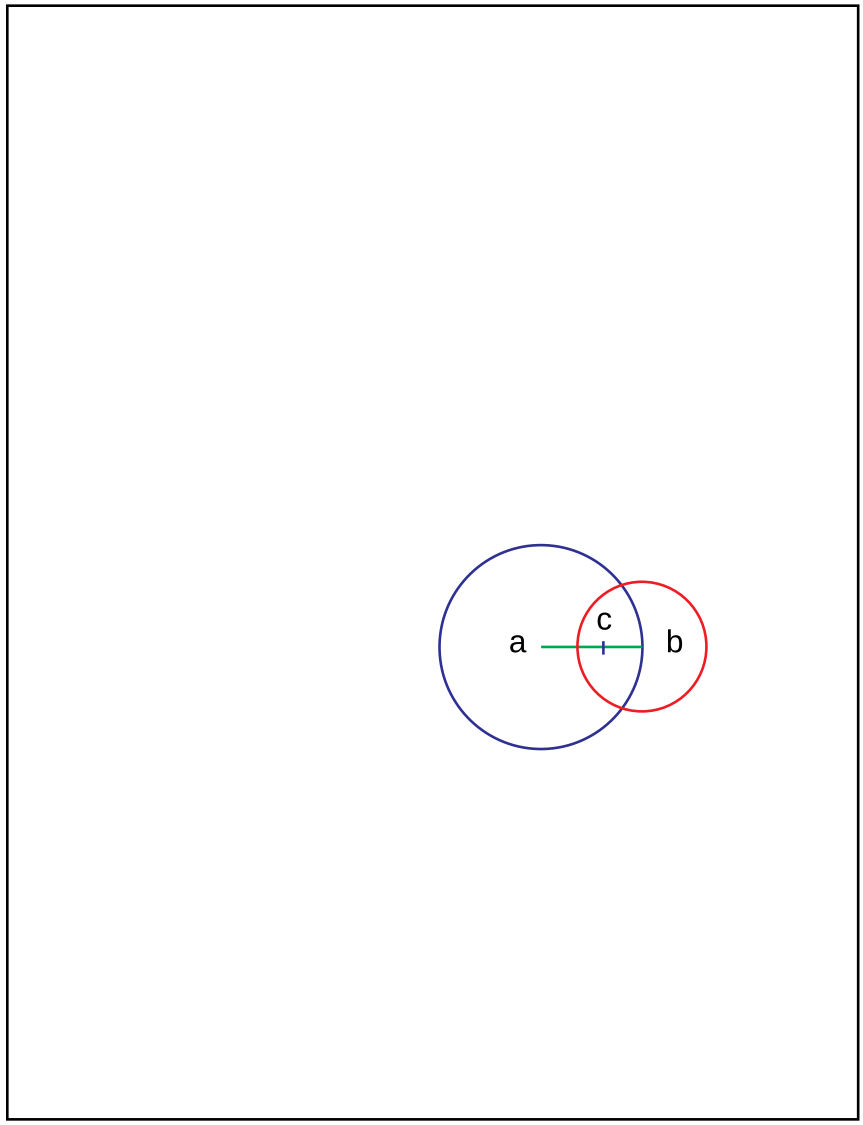
RETURN to IV.10:9.

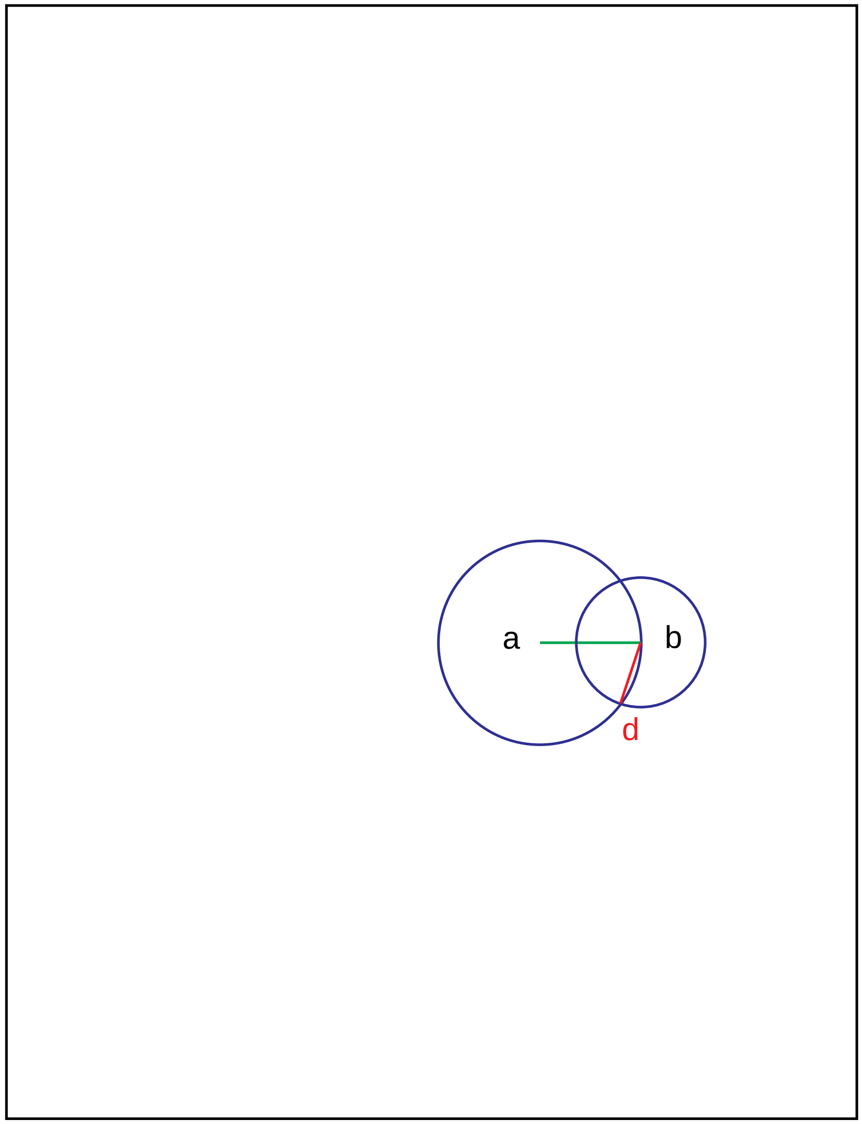


IV.10:14. Let ad be joined.

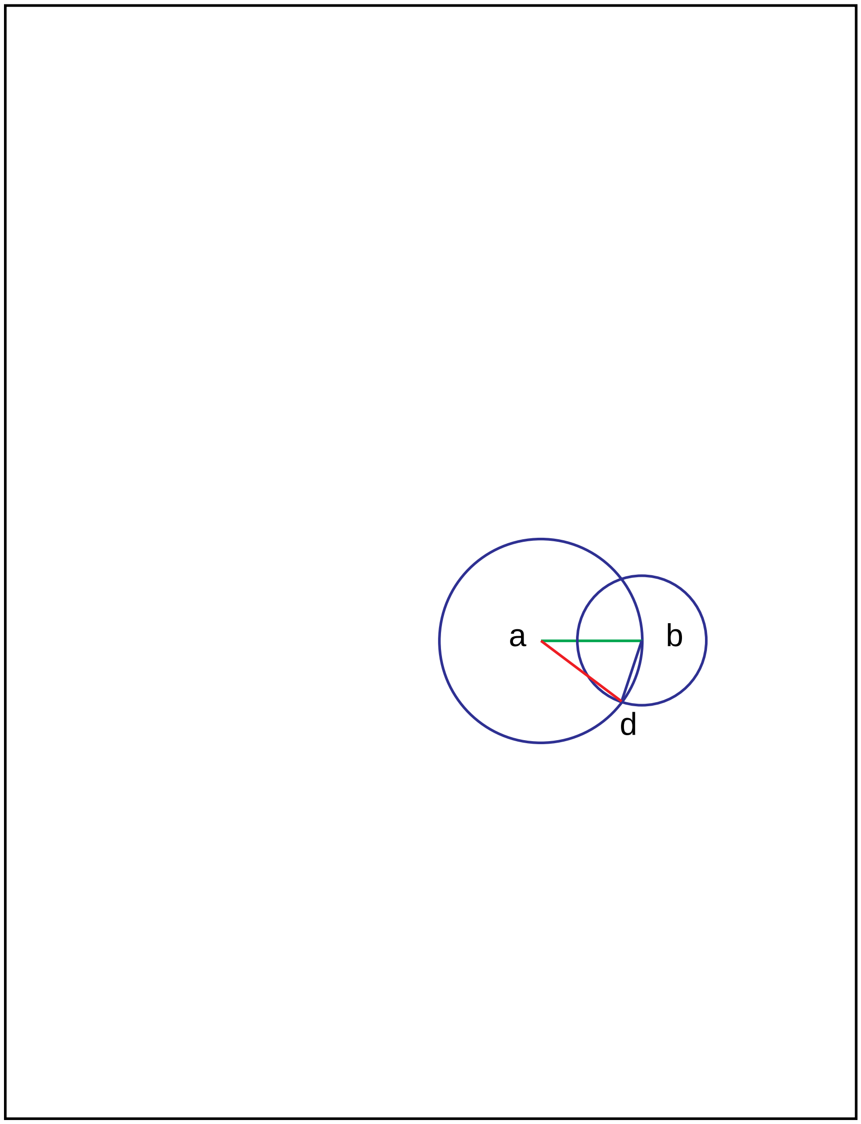


Cleanup.
RETURN from IV.10 t IV.11:6.
Relabel.

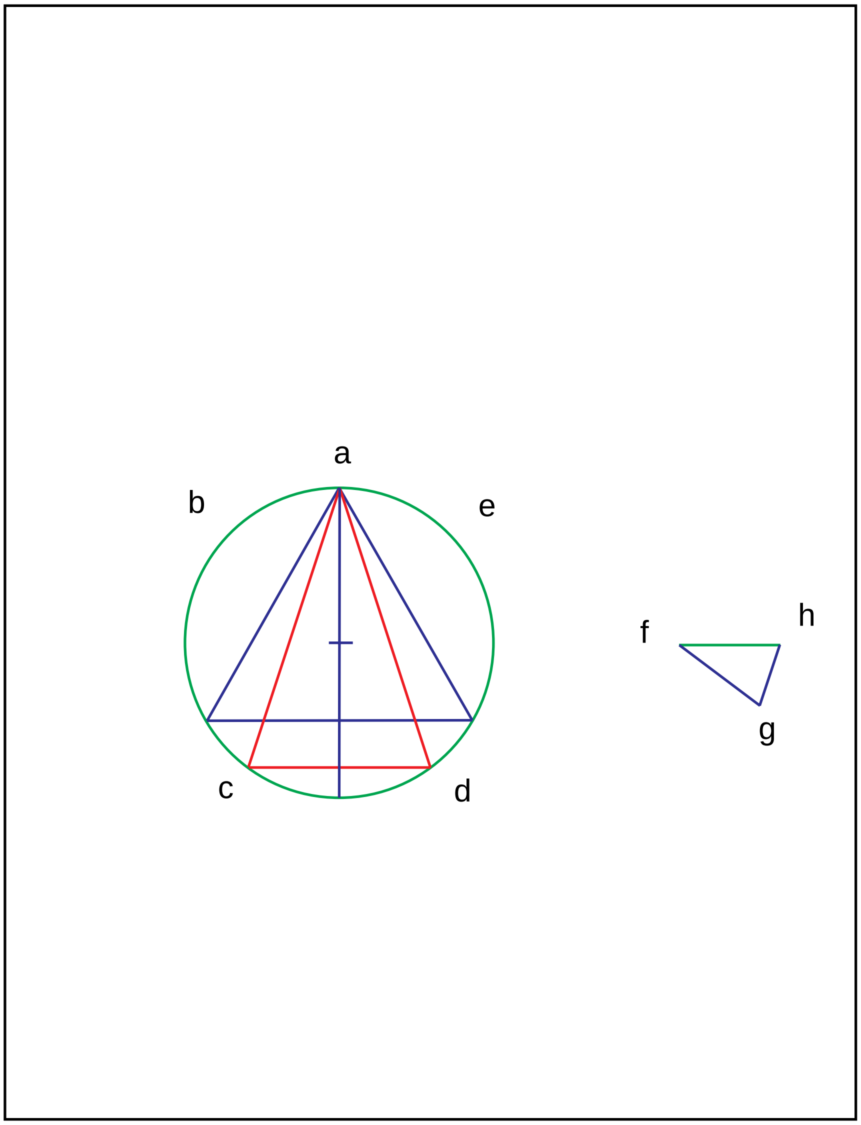

c
a
b


a
b
d

Cleanup.
RETURN from IV.10 to IV.11:6.
Relabel.



IV.11:10. let there be inscribed in the circle abcde the triangle acd equiangular with the tiriangle fgh ...; [IV.2]

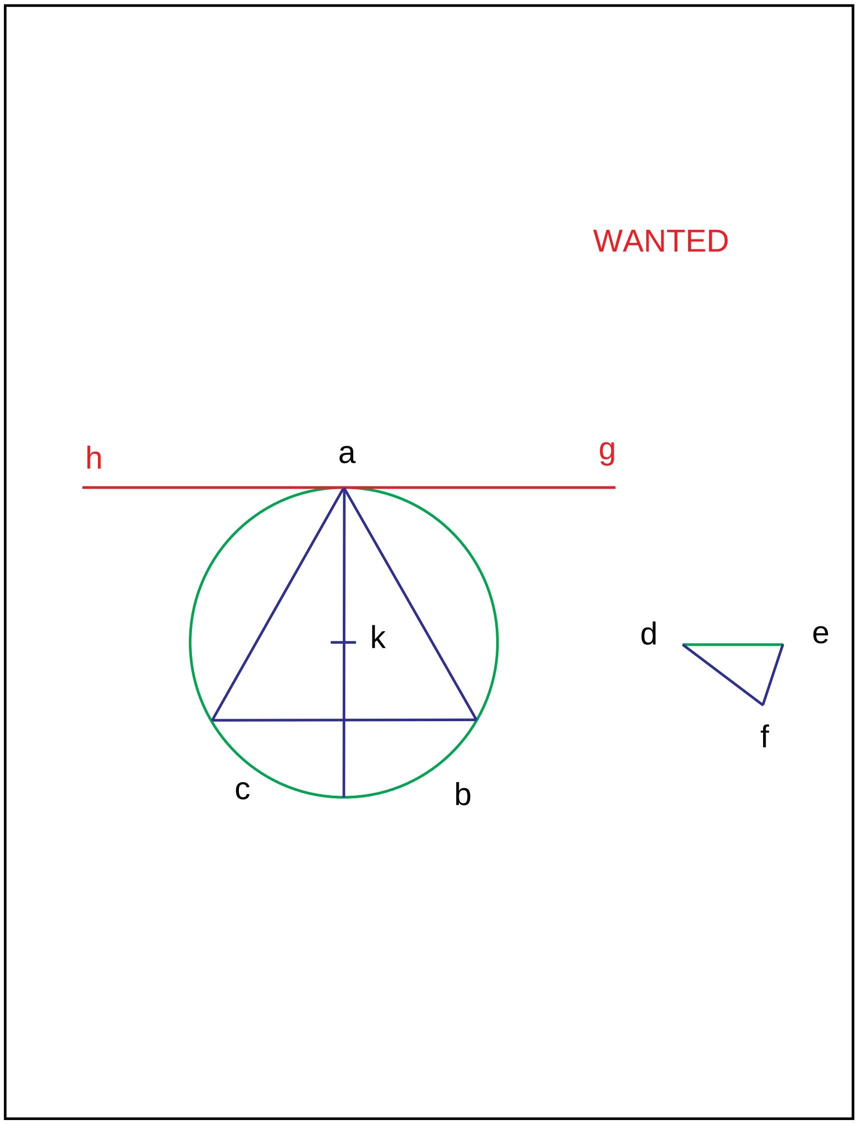


NOTE: The isoceles triangle acd is bisected by the diameter at a, constructed above.

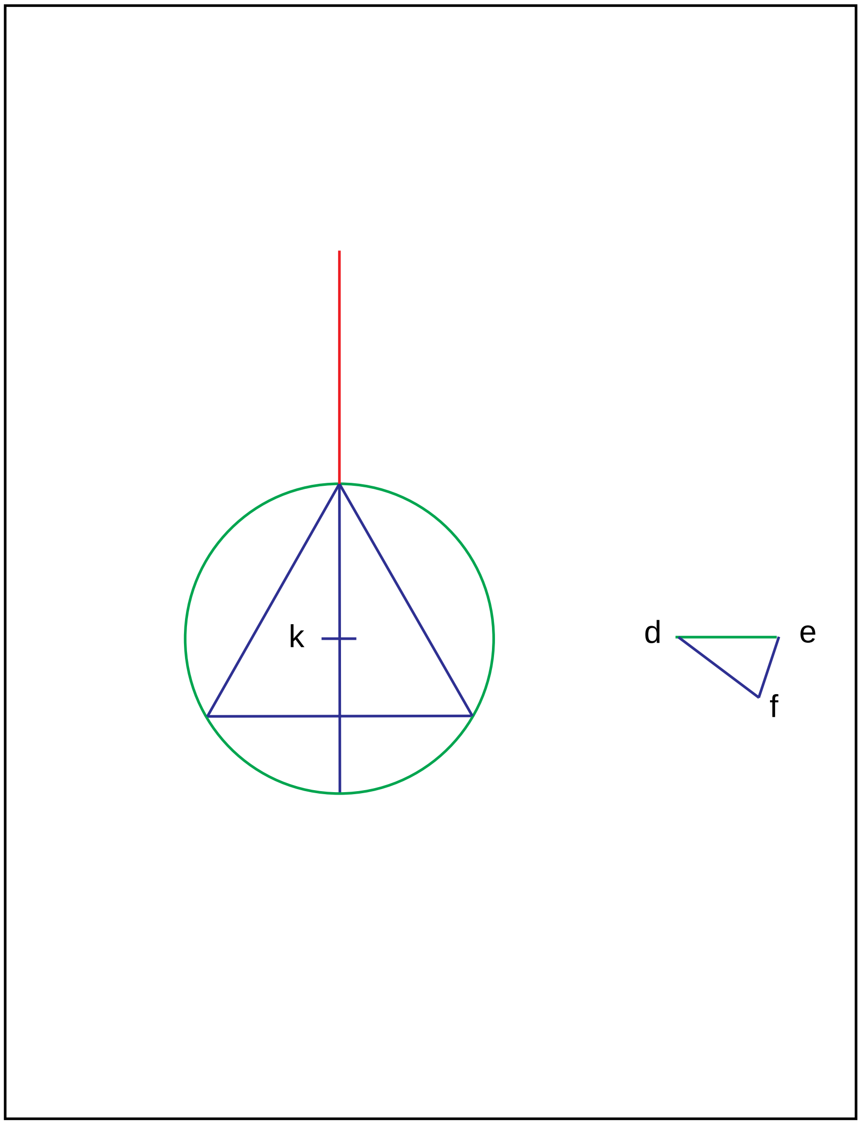
GOSUB IV.2.
Relabel.

IV.2:7. Let gh be drawn touching the circle abc at a. [III.16, Por.] (C#18B)

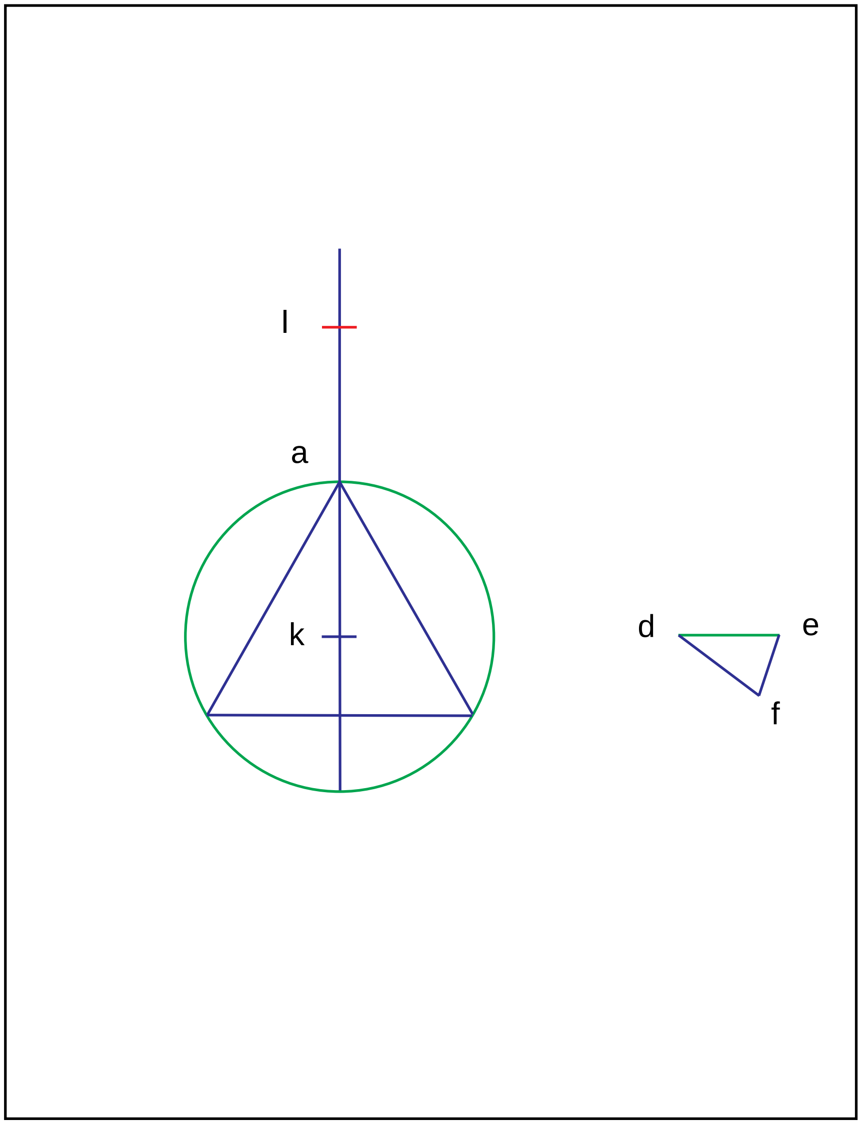
GOSUB C#18B. No relabels. Here we save some steps by recyling the hexagon diameter at a.



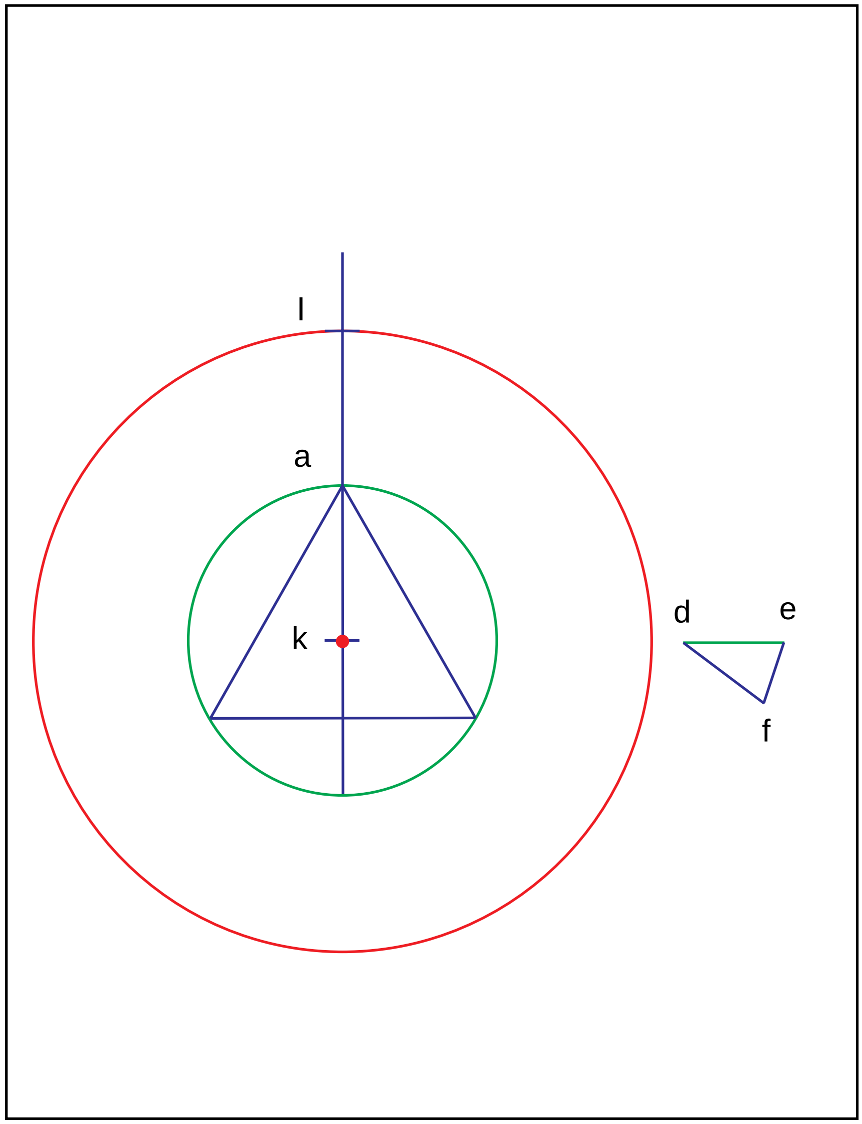
Extend ka upwards.



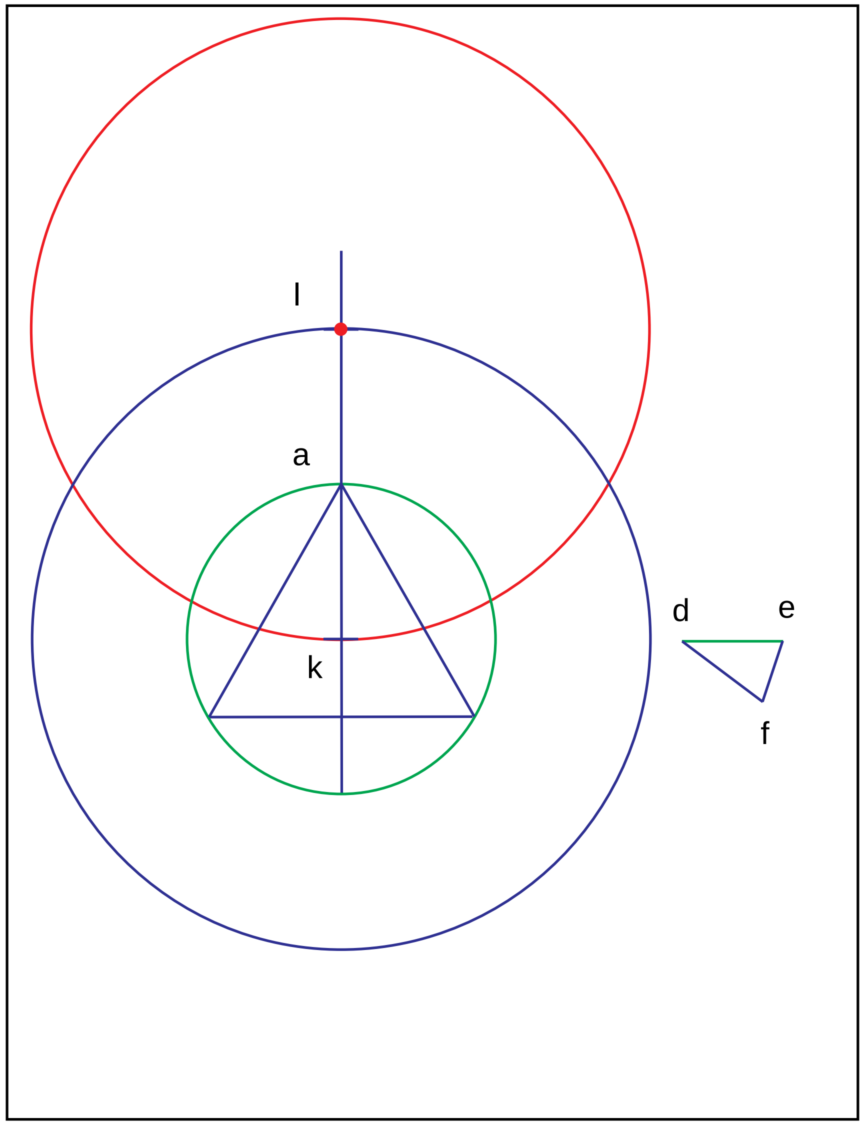
Let al be made equal to ak.



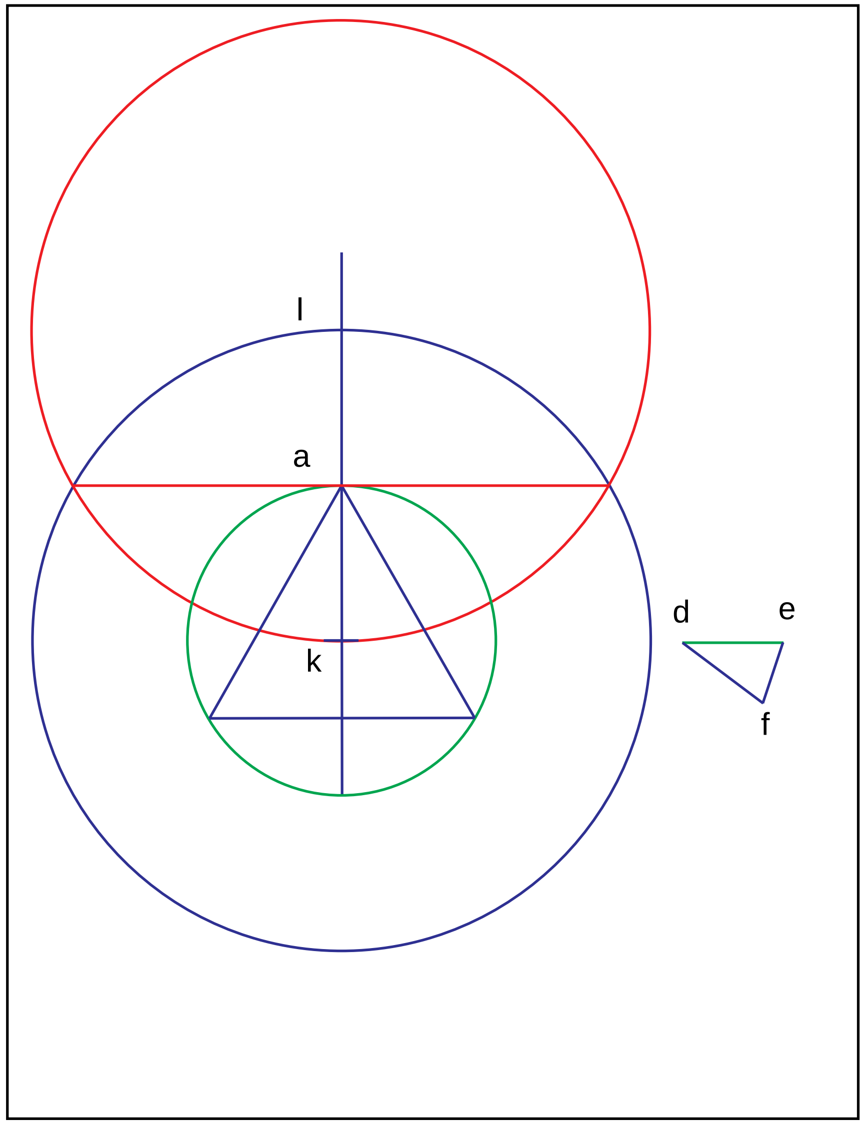
Swing kl around k.



Swing lk around l.

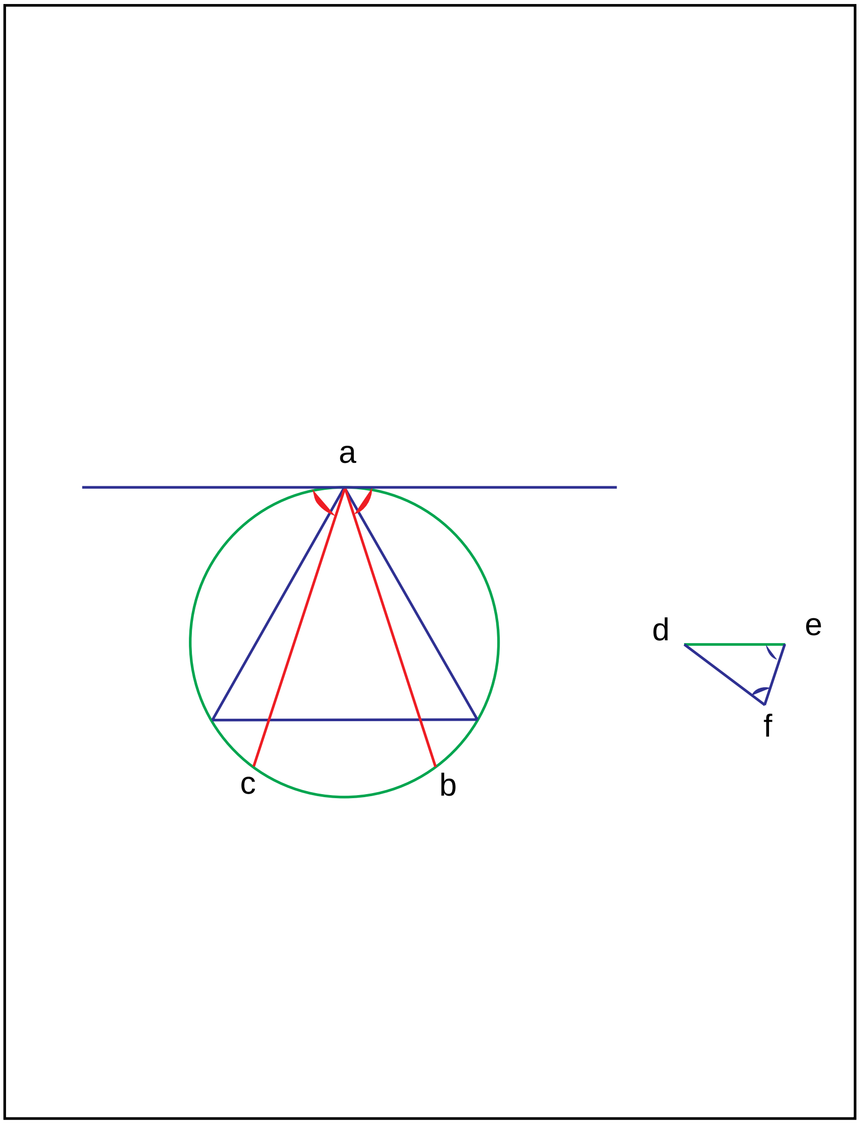


Connect the crossing points.



Cleanup.
RETURN to IV.2:7.
Relabel.

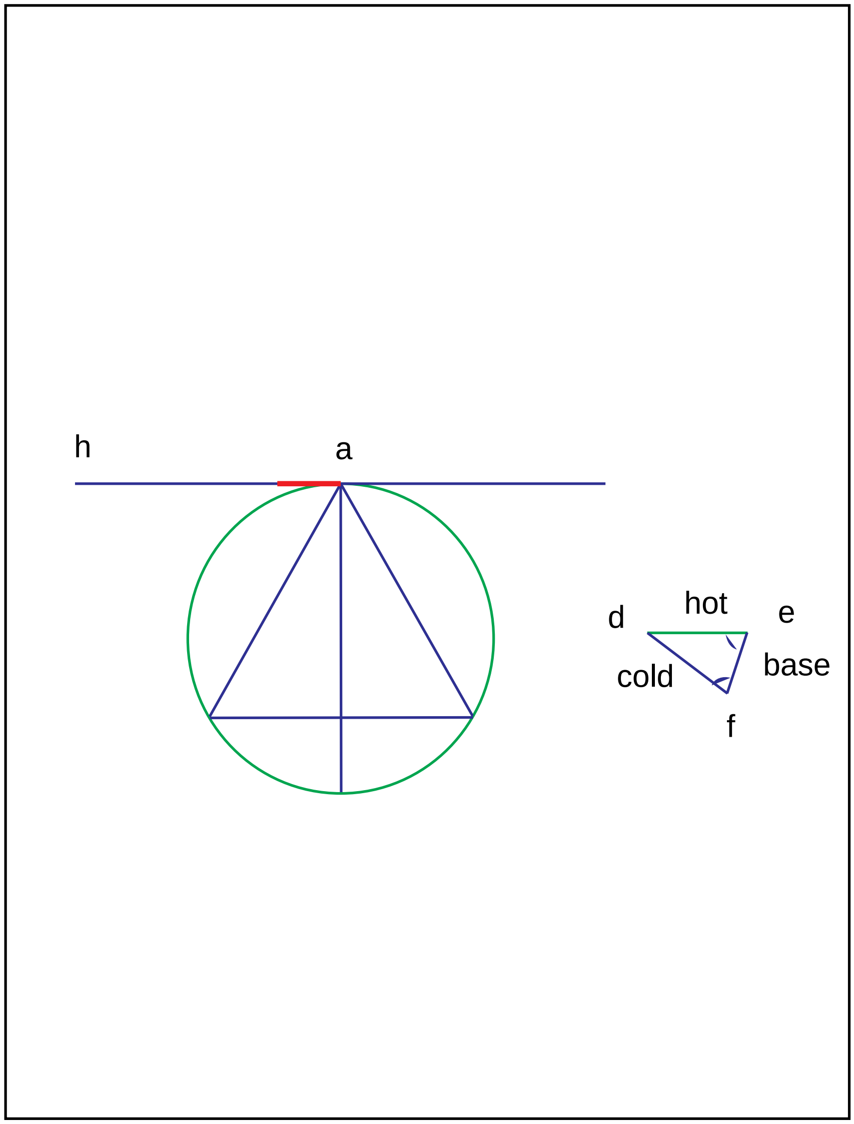
IV.2:8. on the straight line ah, and at the point a on it, et the angle hac be constructed equal to the angoe def, and on the straight line ag, and at the point a on it, let the angle gab be constructed equal to the angle dfe; [I.23]



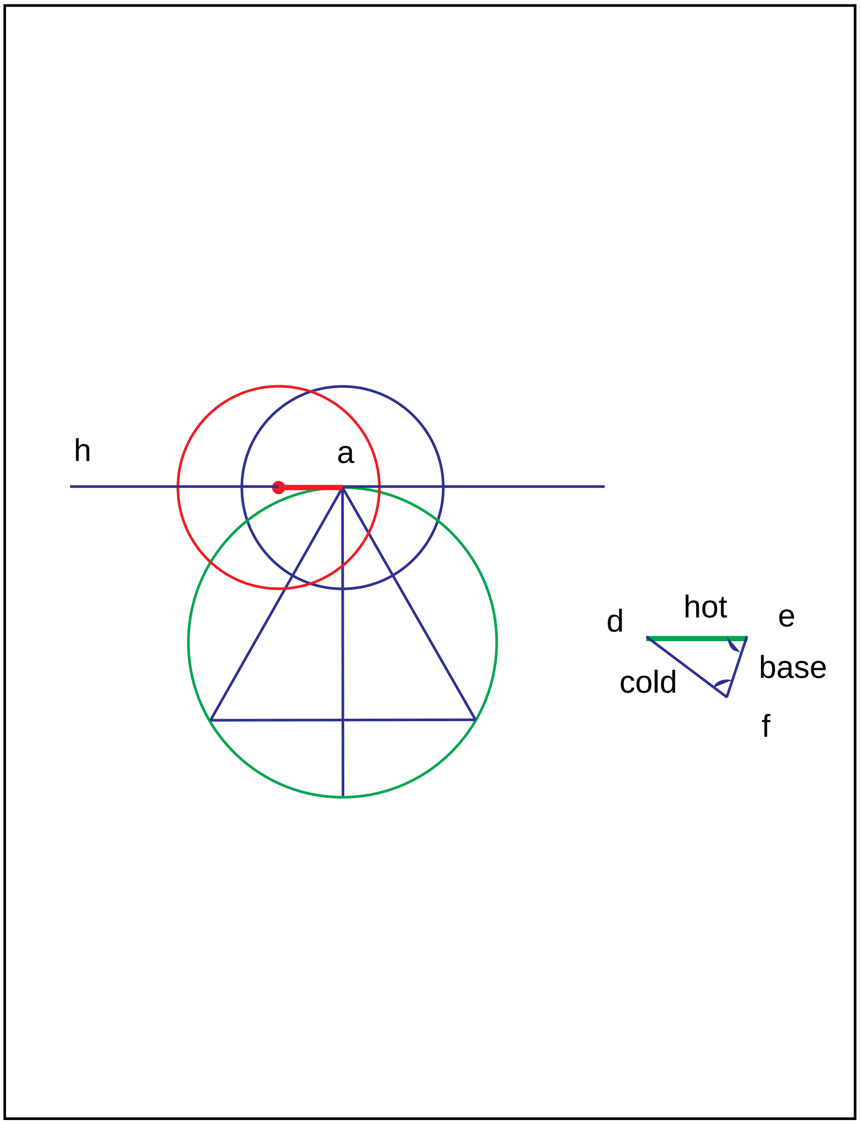
NOTE: These two angles are equal. They are the larger angles of the isosceles triangle def. The smaller angle at d will be moved to a. Again, we can economize.

We follow C#8P, the Proclus variation, choosing ef as the base for both angle constructions.

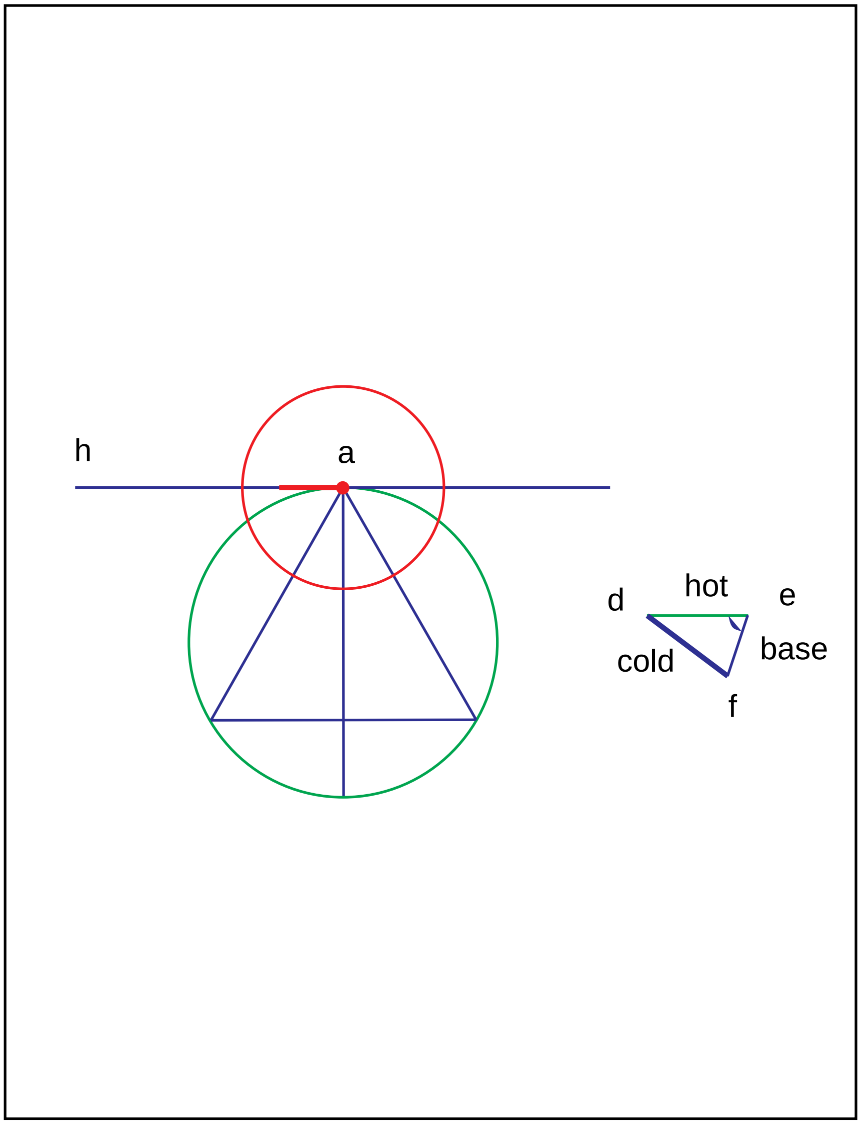
Move the base ef to the line ah.



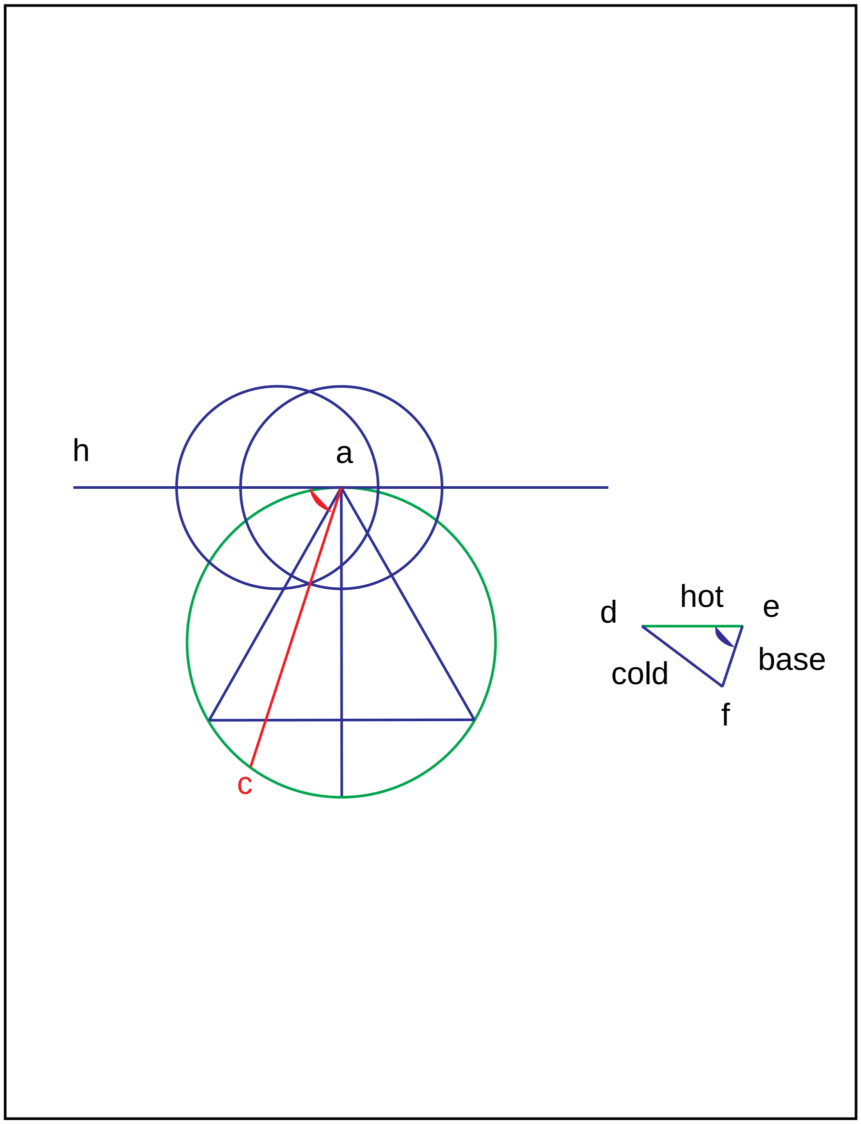
Swing the hot arm around the hot end of the moved base.



Swing the cold arm around the cold end of the moved base.

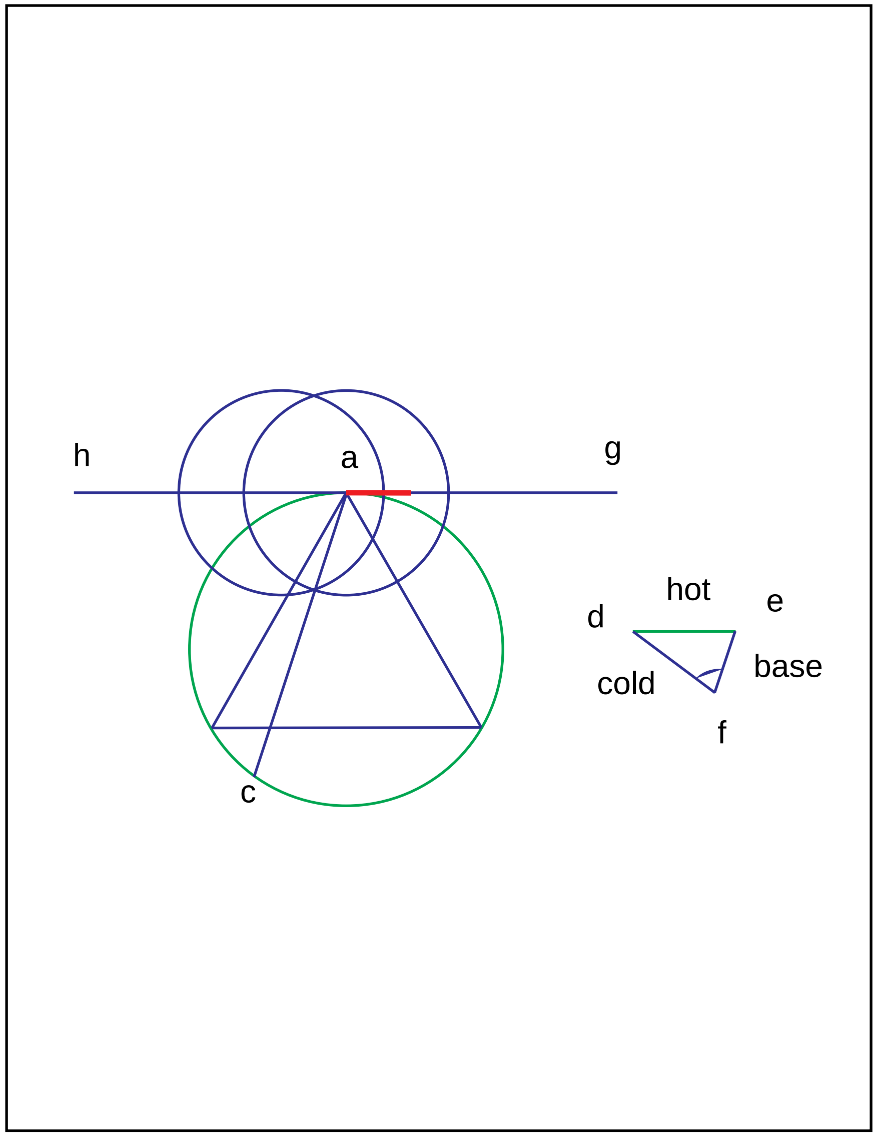


Connect the lower crossing point to the hot end, extend across the given circle, locate the point c.

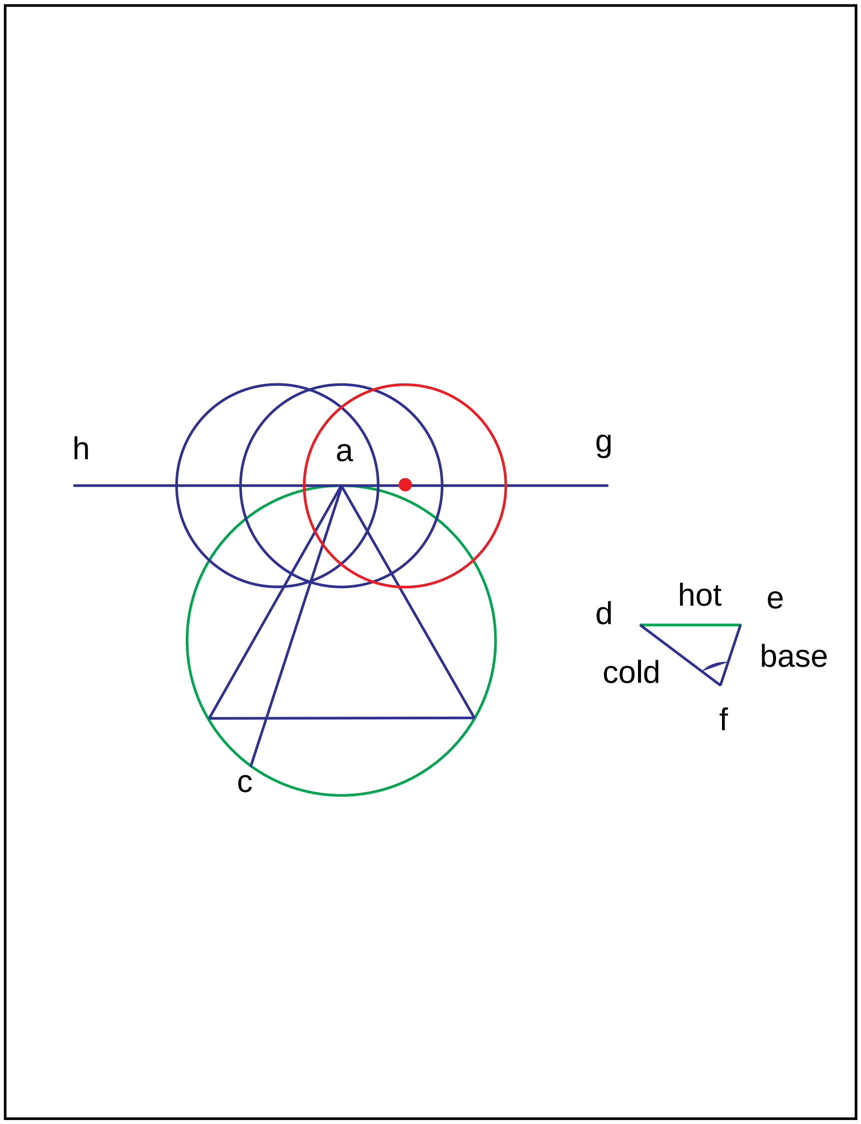


GOSUB C#8P again for the equal angle gab to locate the point b. We must swap the hot and cold labels on the golden triangle def.

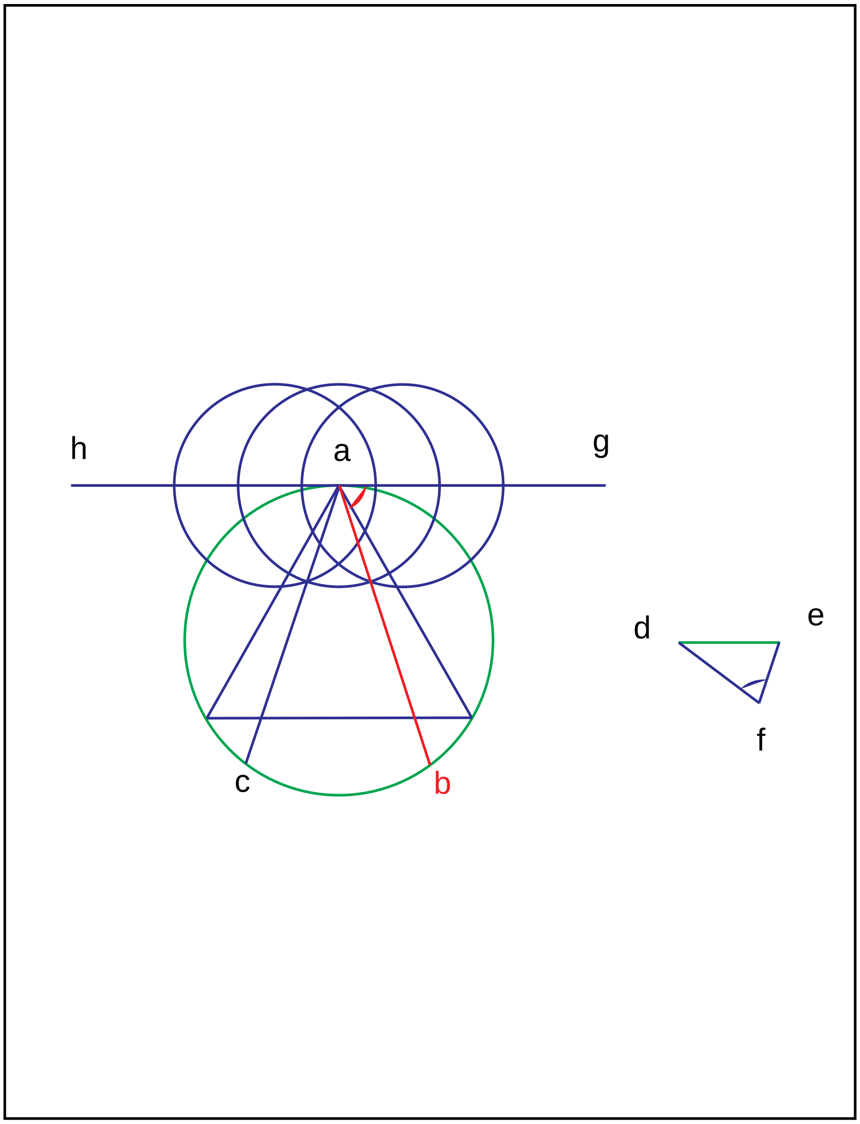
Move the base ef to the line ag.



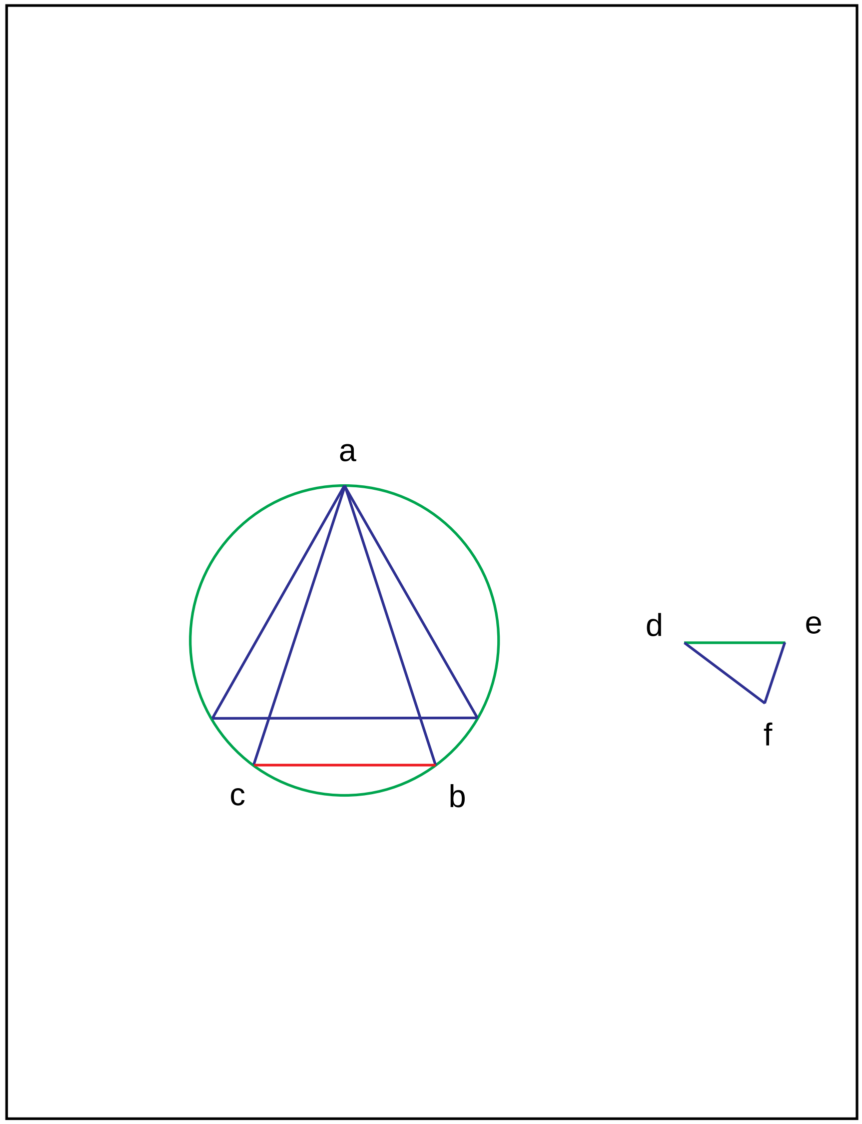
Swing the hot arm around the hot end: we have this circle at a already, as dc and df are equal. Swing the cold arm around the cold end.



Connect the lower crossing point to a, extend across the given circle, locate the point b.

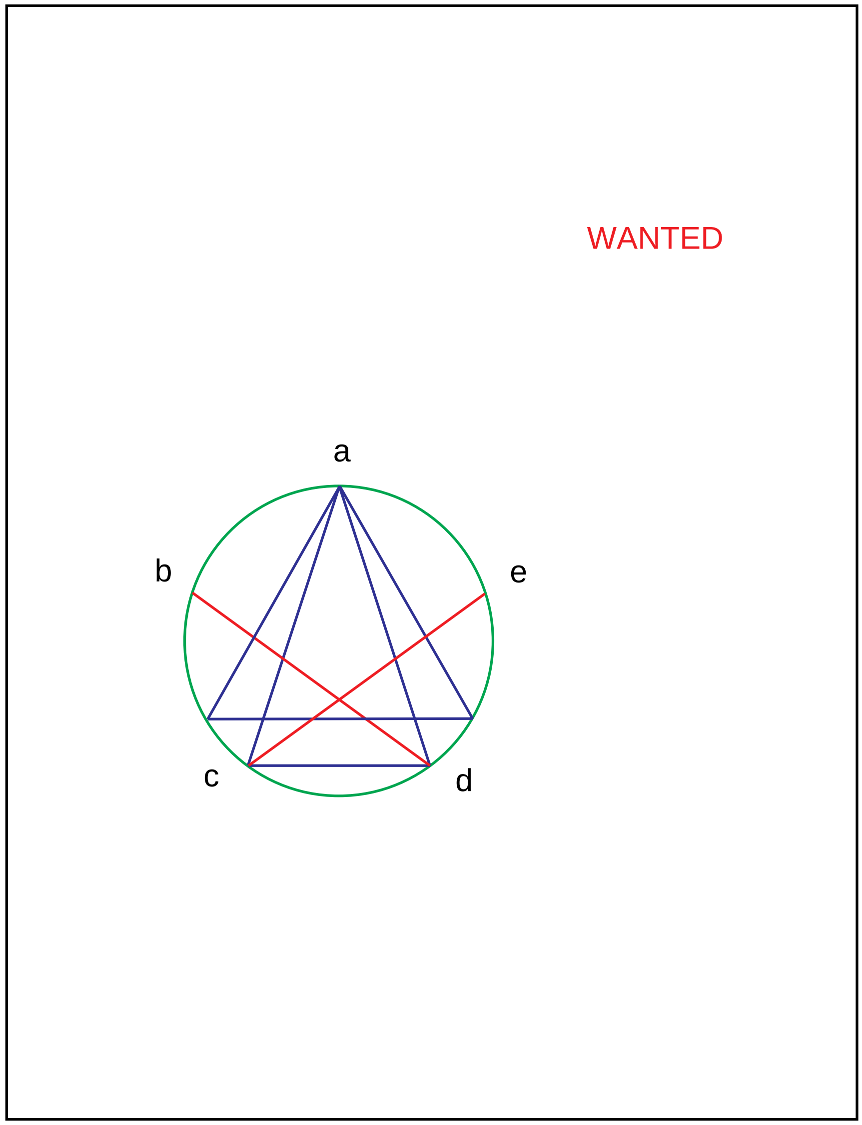


Cleanup.
RETURN to IV.2:8.
IV.2:13. let bc be joined.



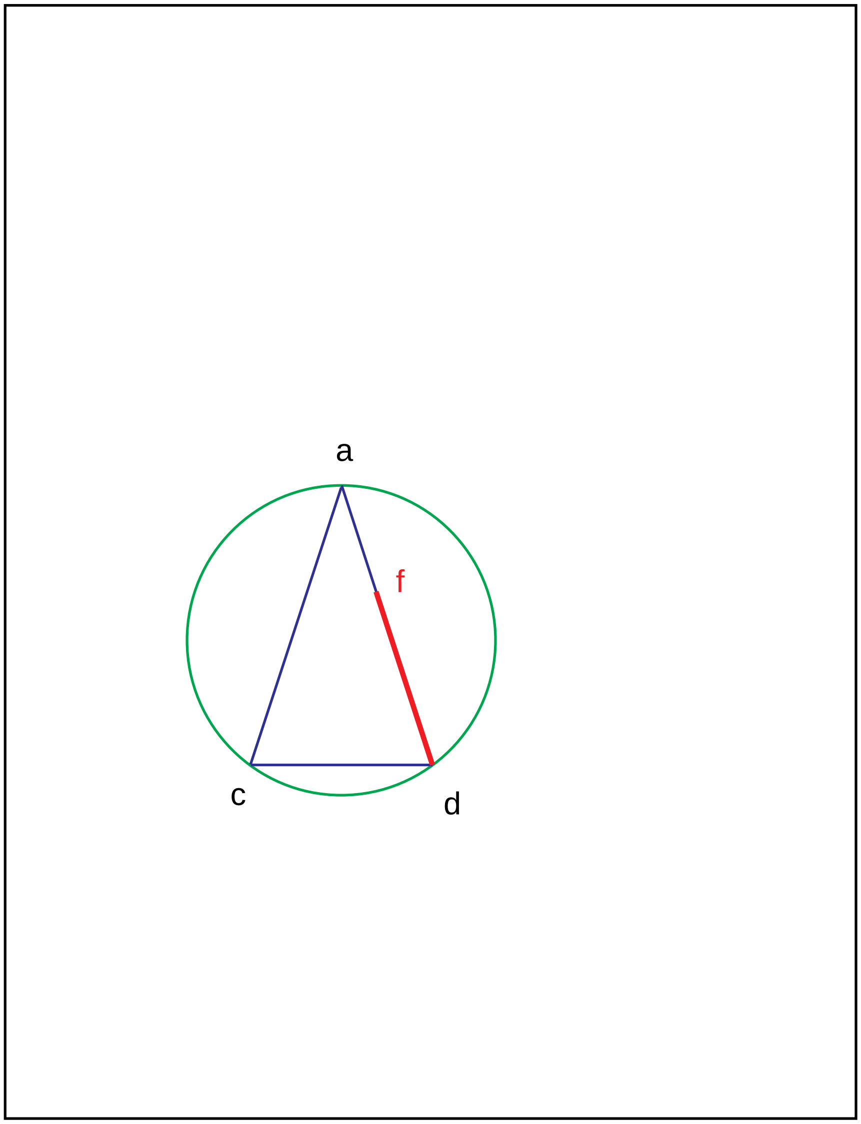
RETURN to IV.11:10.
Relabel.

IV.11:17. Now let the angles acd, cda be bisected respectively by the straight lines ce, db [I.9],

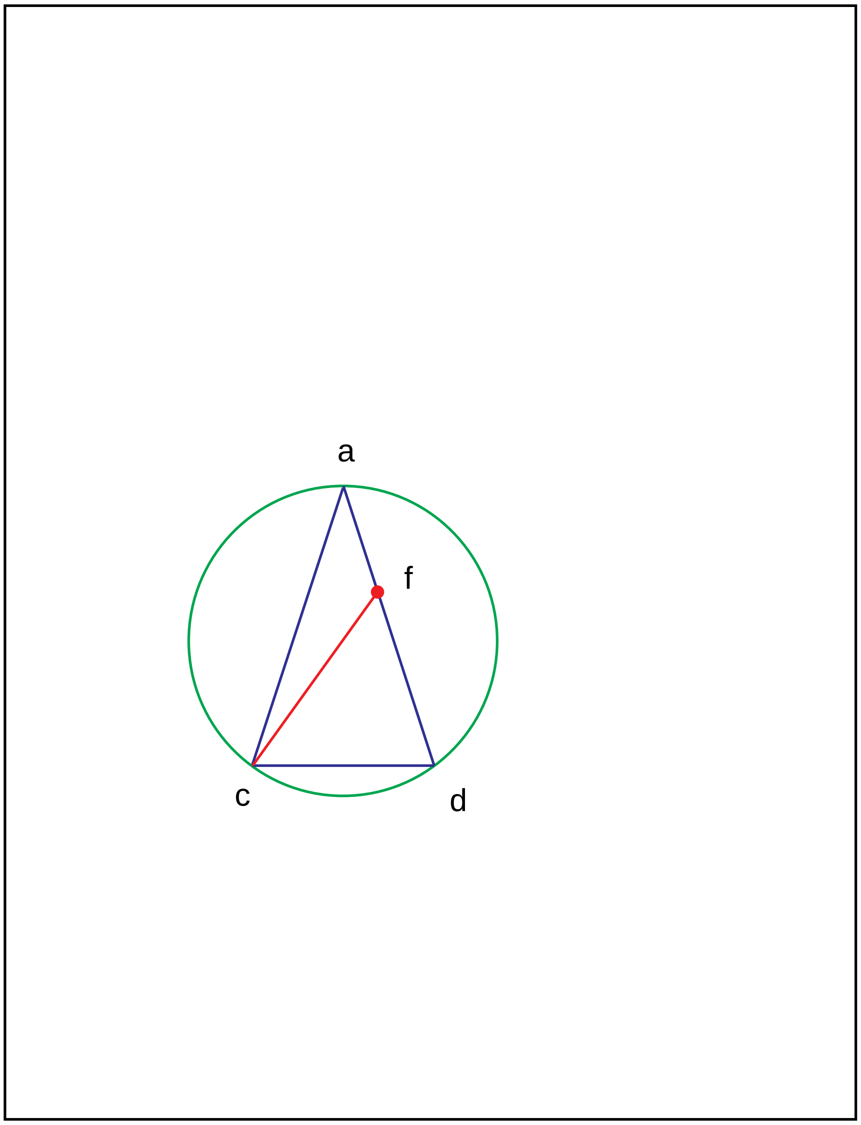


For IV.16, we actually require only one bisection of eda to locate the point b.

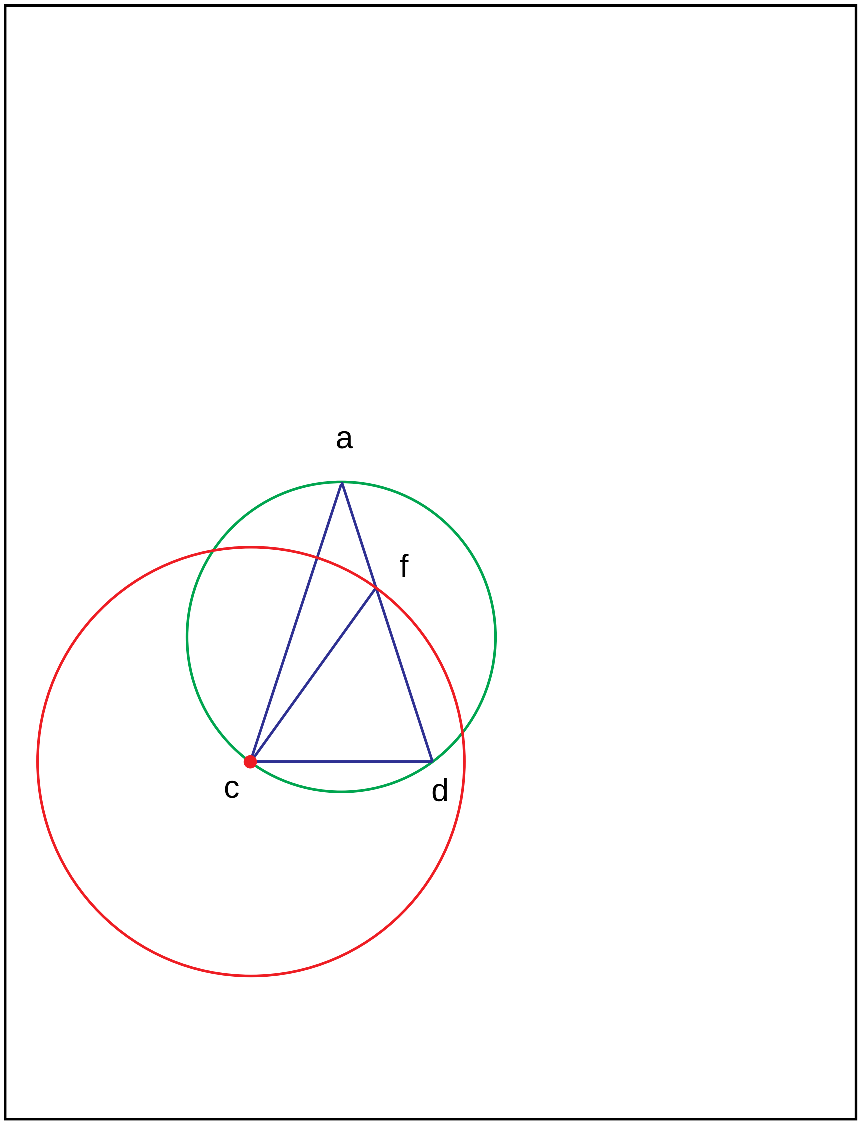
GOSUB I.9.
No relabel.
I.9:5. let df be cut off from da equal to dc; [I.3]



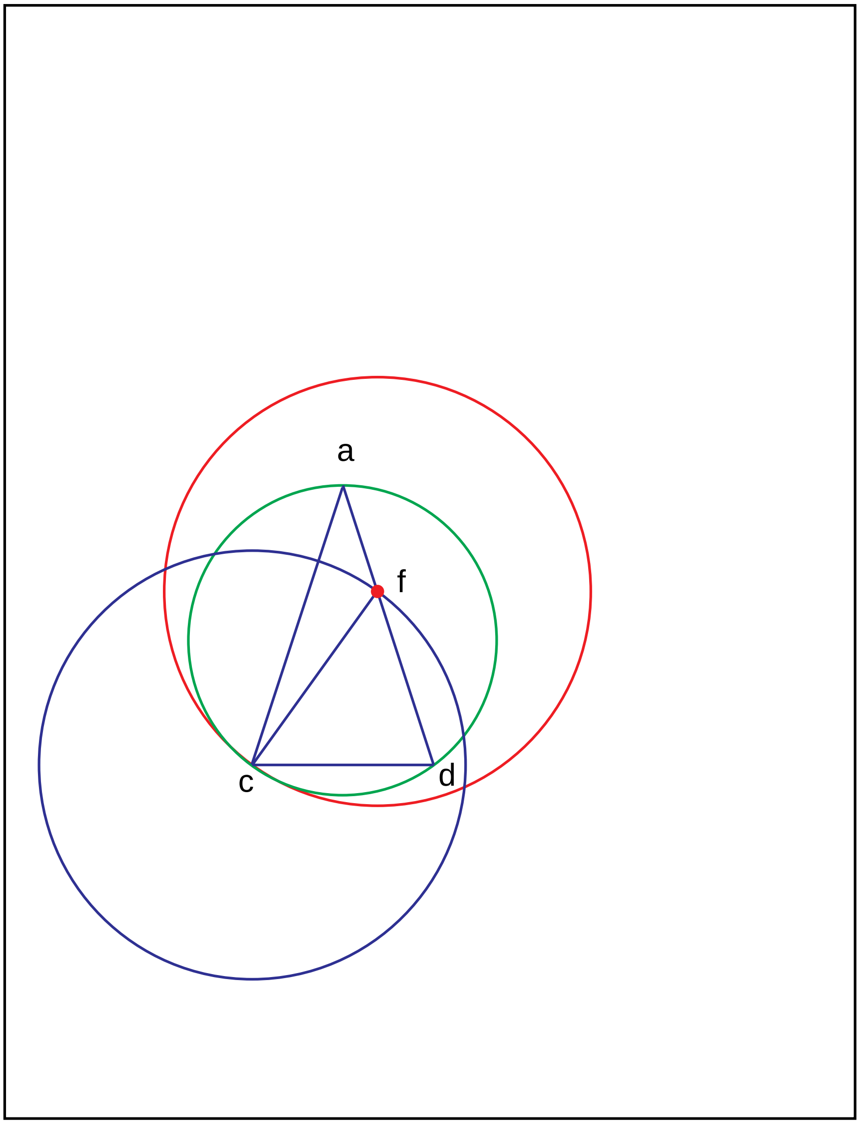
I.9:6. let cf be joined,



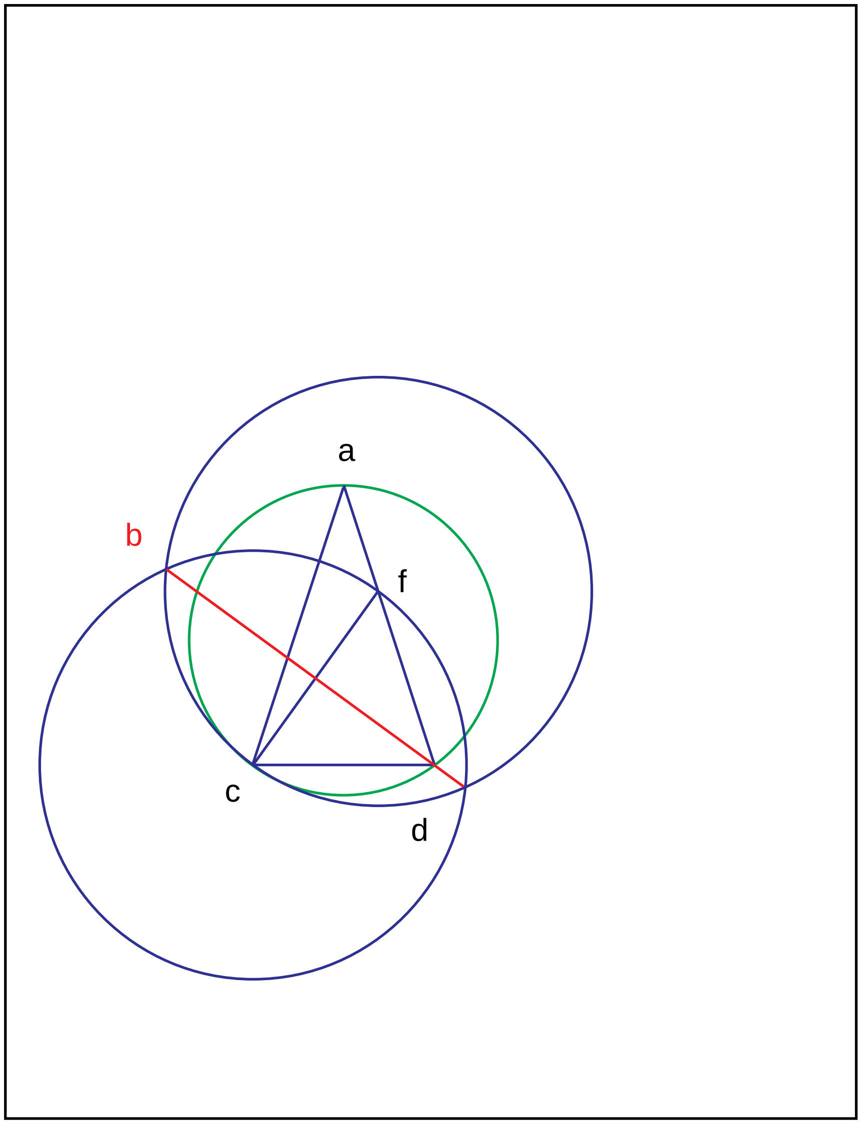
I.9:6. and on cf let the equilateral triangle efg be constructed; ([I.1])
GUSUB I.1.
Swing cf around c.



Swing fc around f.



Connect the crossing points.
Locate the point b.



Cleanup.

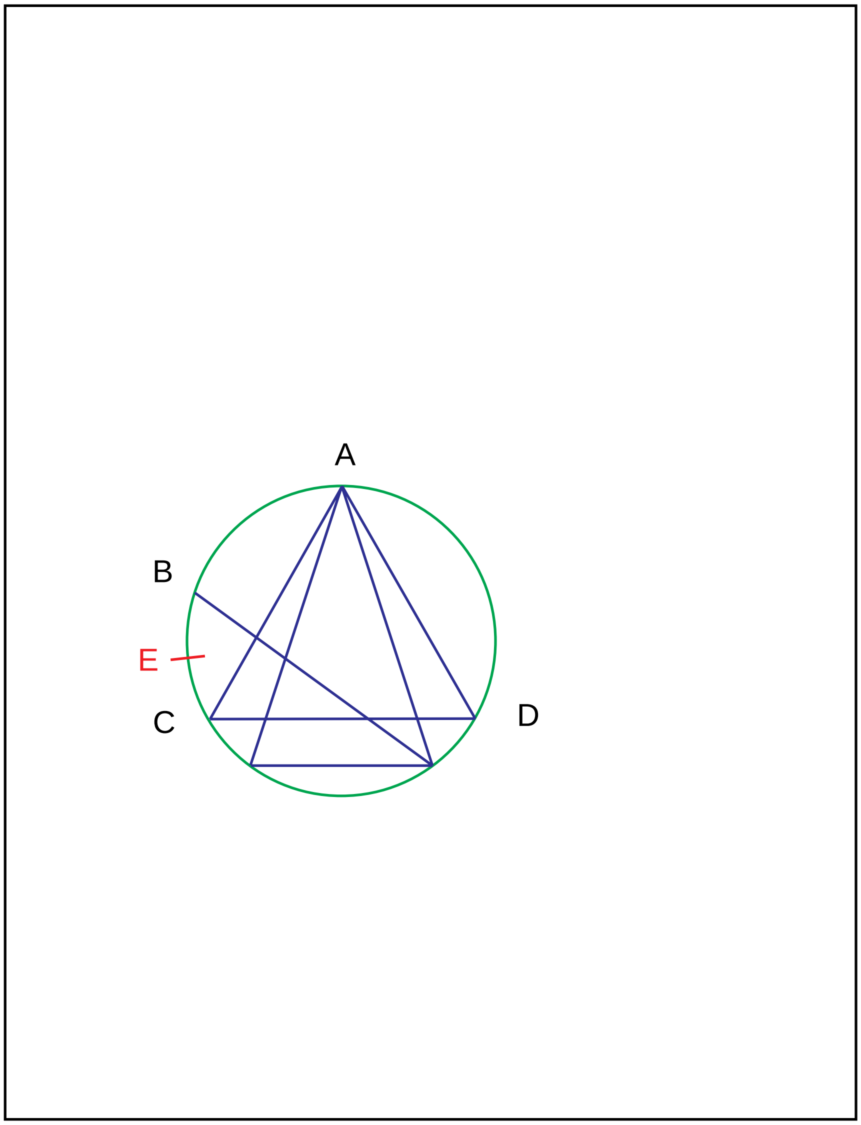
RETURN to I.9:6.

RETURN to IV.11:17.

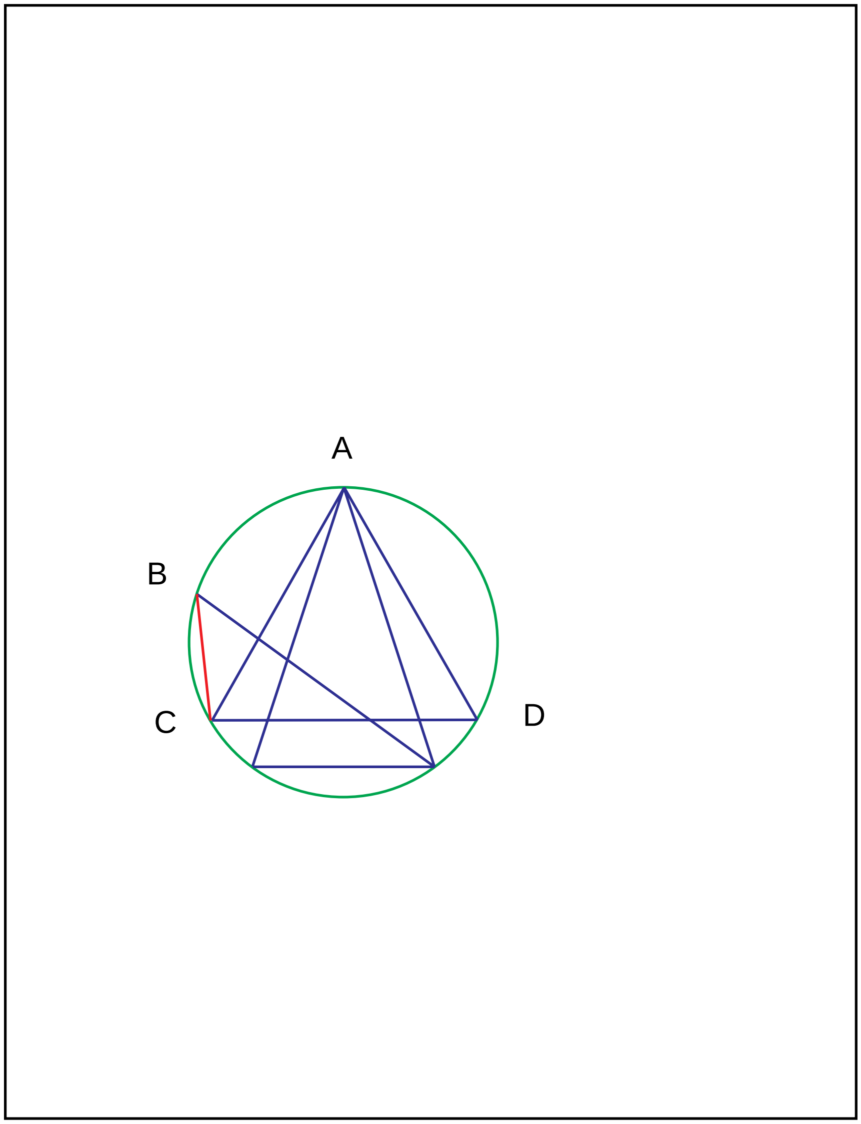
RETURN to IV.16:11.
Relabel.

IV.16:23. Let BC be bisected at E. [III.30]

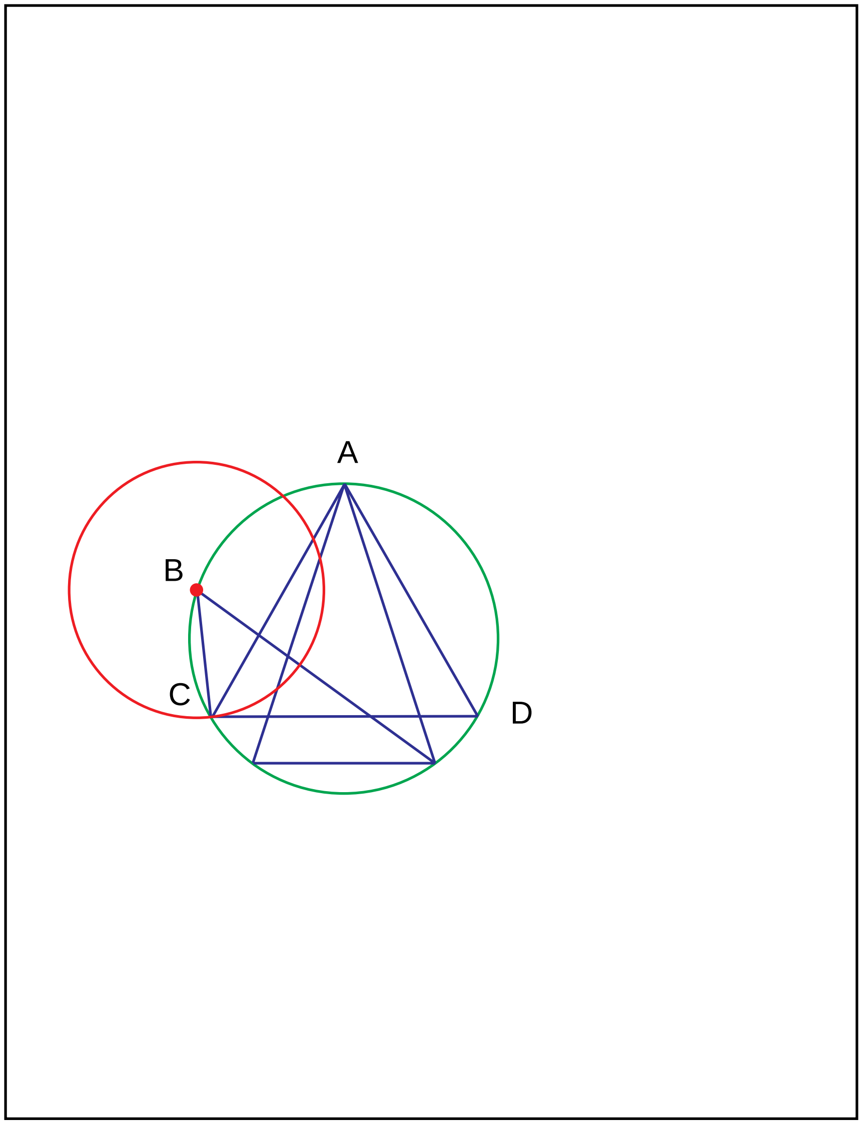
GOSUB III.30, C#20.



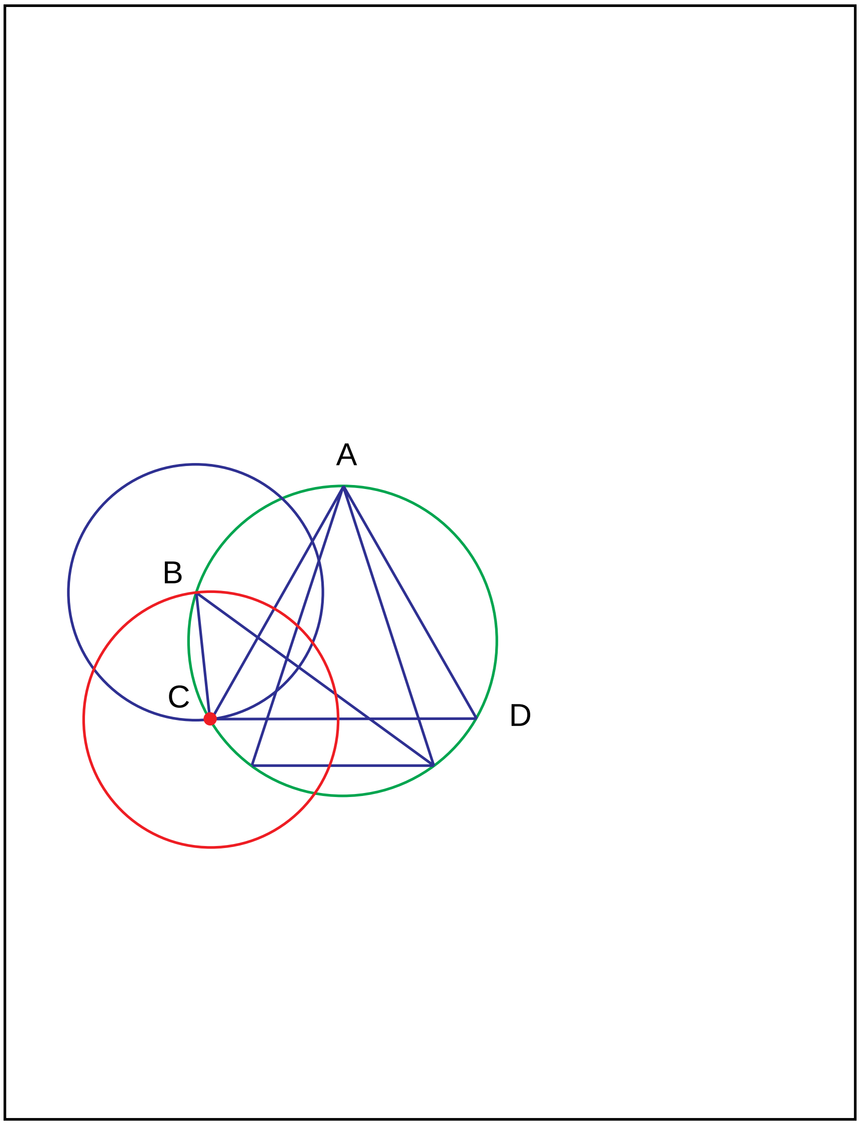
III.30:4. Let BC be joined and bisected at E;
([I.10], C#5B)



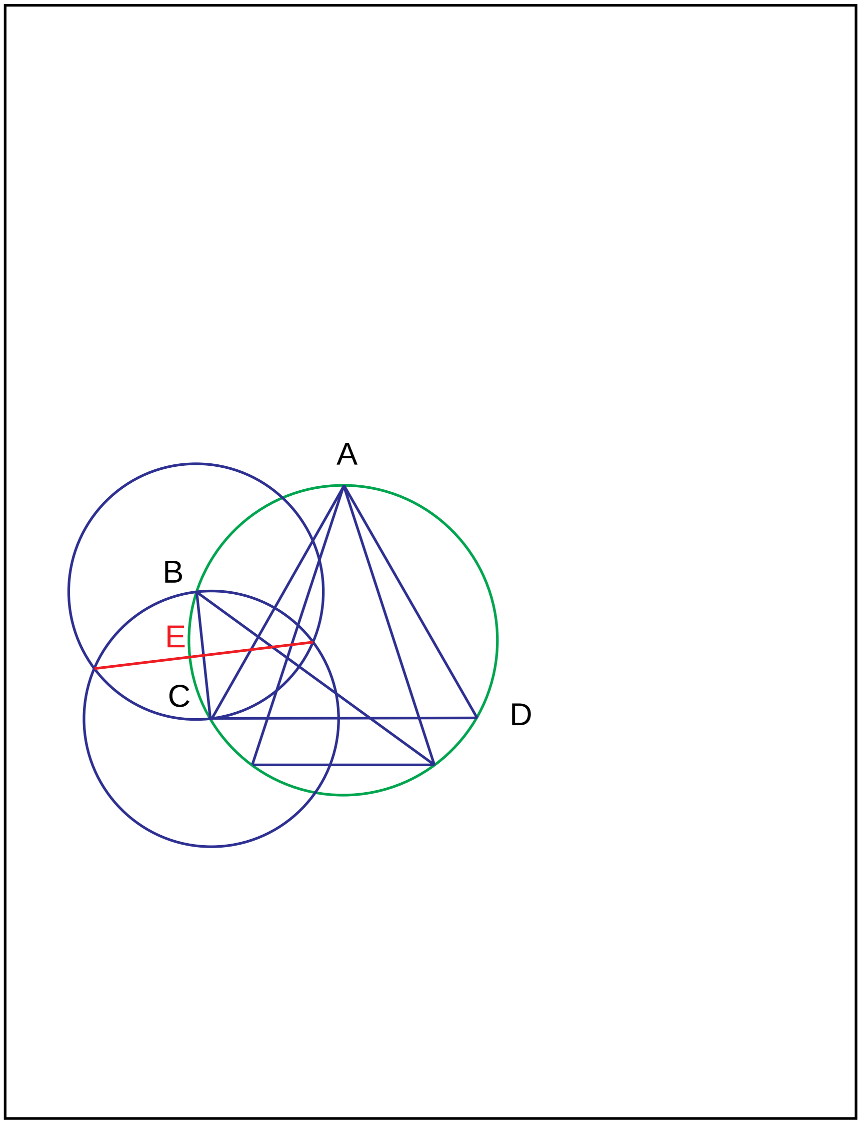
Swing BC around B.



Swing CB around C.

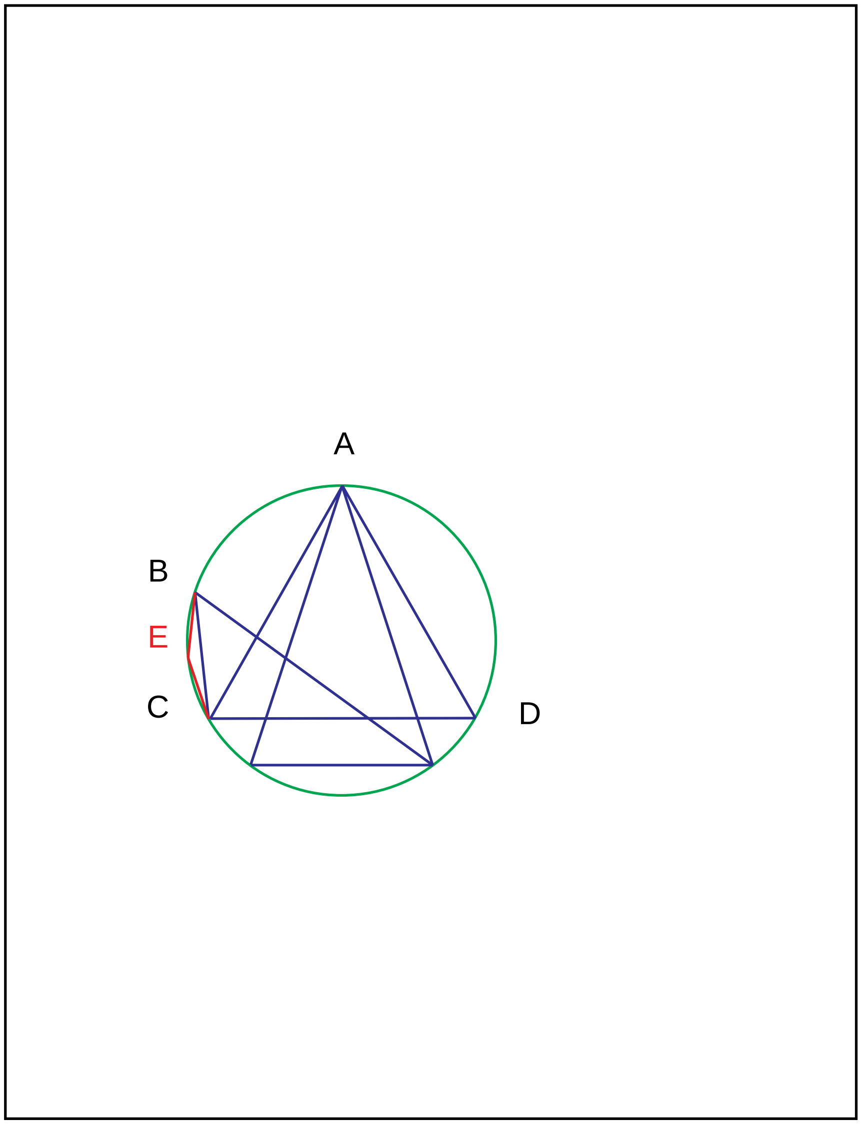


Connect the crossings. Locate the point E on the circumference of the circle ABCD.



Cleanup. RETURN to III.30:4.
RETURN to IV.16:23.

IV.16:26. If therefore we join BE, EC and fit into the circle ABCD straight lines equal to them and in contiguity, a fifteen-angled figure which is both equilateral and equiangular will have been inscribed in it.



Cleanup.
DONE.

NOTE: 1) Now that we see how Euclid finishes this construction, we see that the fifteenth part of the circle was available to us after step 38, as the adjacent vertices of the inscribed pentagon (golden triangle) and equilateral triangle are before us there. The remaining steps then are required only for the proof. Thus, we may regard C#38 as a 38-step construction. In fact, even for the proof, this last subroutine, C#20, is unnecessary, as here we have constructed the hexagon instead of Euclid's direct construction of the equilateral triangle. The construction of an inscribed equilateral triangle from the hexagon could be set out as a Porism after IV.15. The proof could equally have been given using only the inscribed hexagon and pentagon. So step 12 is unnecessary. After 38 steps we have the angle of 24 degrees which is the astronomical motive, according to Proclus. See Heath v2 p. 111.