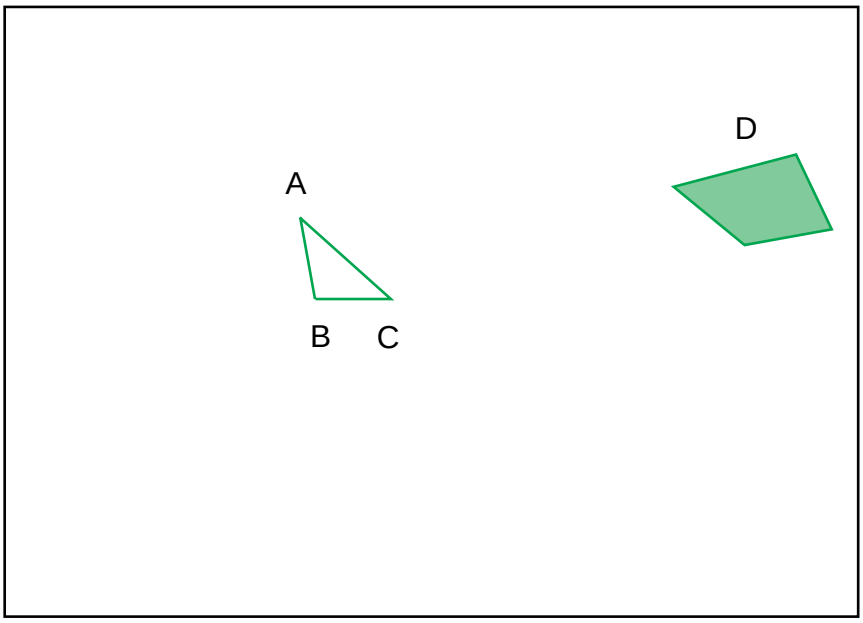
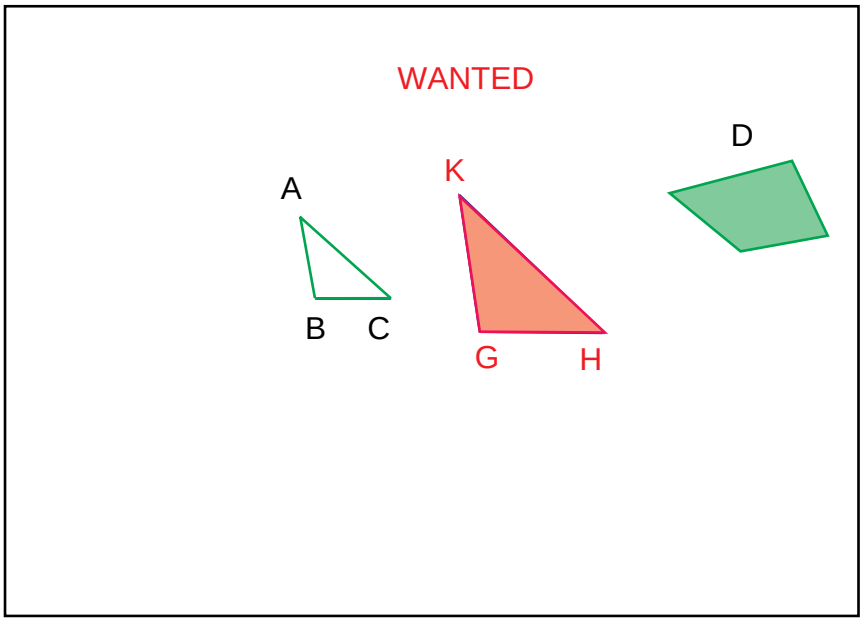
Construction 45: Book VI, Proposition 25

VI.25.1. To construct one and the same figure similar to a given rectilineal fiigure and equal to another given rectilineal figure.

VI.25:3. Let ABC be the given rectilineal figure to which the figure to be constructed must be similar, and D that which it must equal,



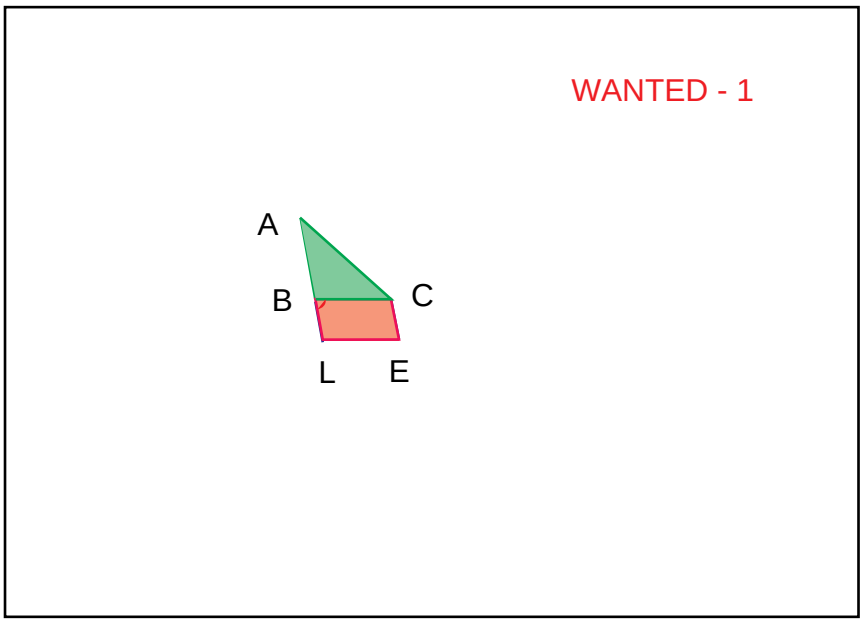
VI.25:6. Thus it is required to construct one and the same figure similar to ABC and equal to D.



As this is a compound construction, using the longest subroutine of Book I, we may break it into stages, following Euclid's program. Within the stages we will economize the steps. When we see a shortcut, we will take it.

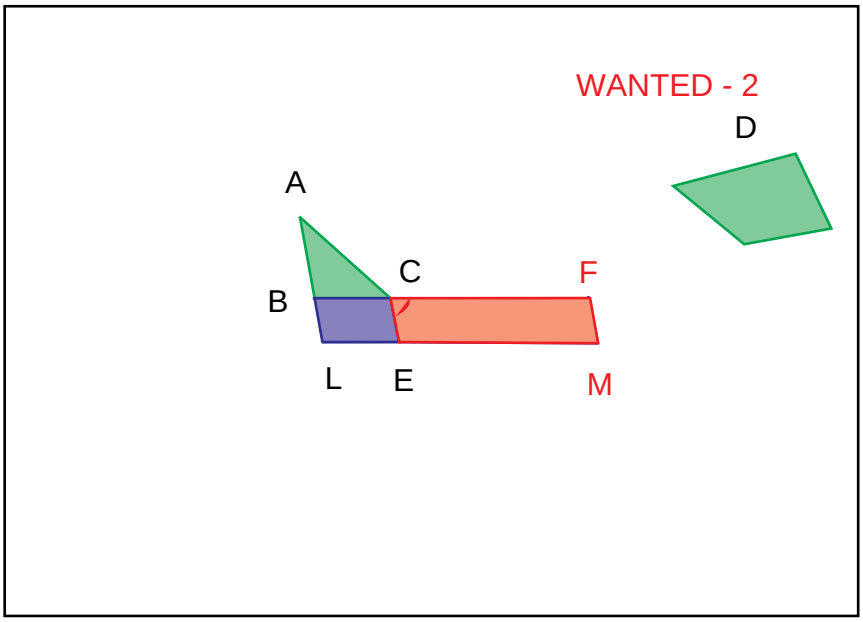
Stage 1.

VI.25:8. :Let there be applied to BC the parallelogram BE equal to the triangle ABC [I.44],



Recall that I.44, C#12A or C#12B, constructs the parallogram BE equal to ABC in a given angle CBL. Here we are given no angle, and we may choose any convenient angle. The figure above is taken exactly from Euclid, but we will find it economical to orient BL inline with AB, shown below.
Stage 2.

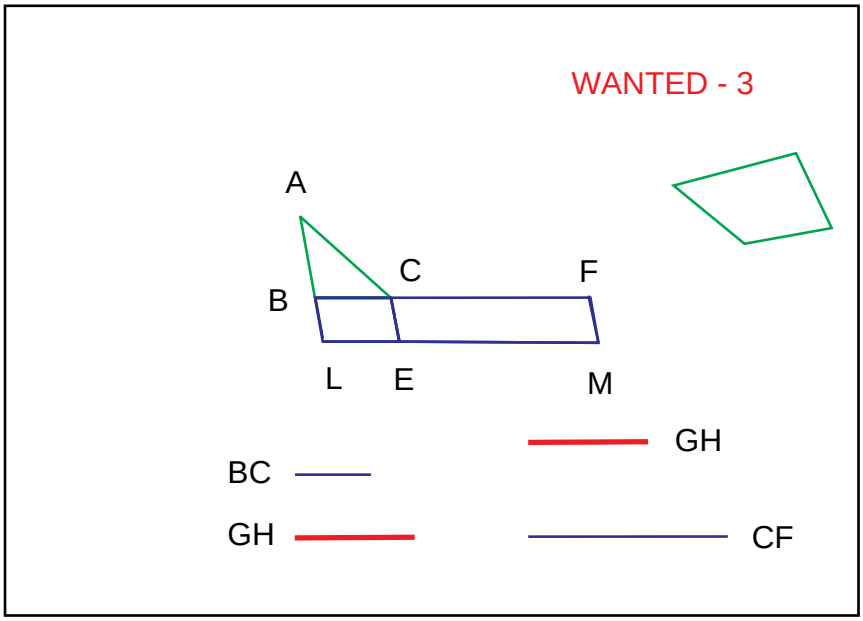
VI.25:9. and to CE the parallelogram CM equal to D in the angle FCE which is equal to the angle CBL. [I.45]



Recall that C#12B < I.44 has 31 steps, while C#13 < I.45 consists of C#12B applied repeatedly, once for each triangle of the figure D. In this special case, our figure D is made of two triangles. Thus, stages 1 and 2 together may take over 90 steps!

Stage 3.

VI.25:14. Now let GH be taken a mean proportional to BC, CF [VI.13],

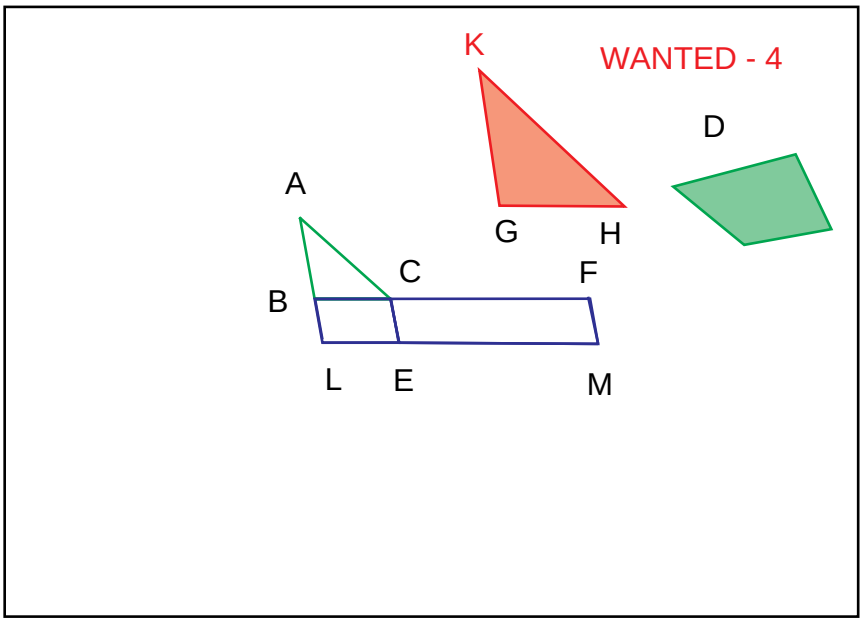


Stage 4.

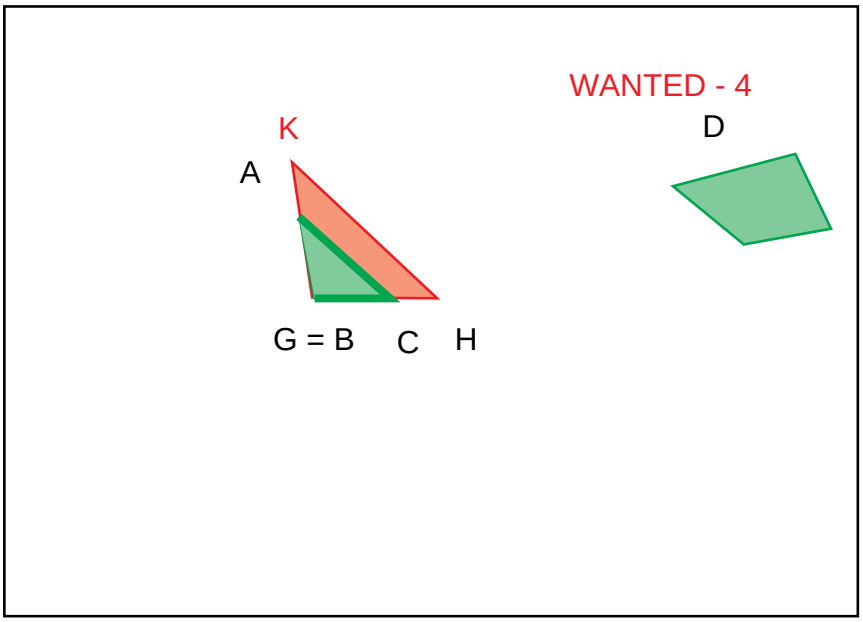
VI.25:15. and on GH let KGH be described similar and similarly situated to ABC. [VI.18]

DONE

As VI.13 (C#43, 9 steps) and VI.18 (C#44, 13 steps) are relatively short, we will closely follow Euclid in these final stages.

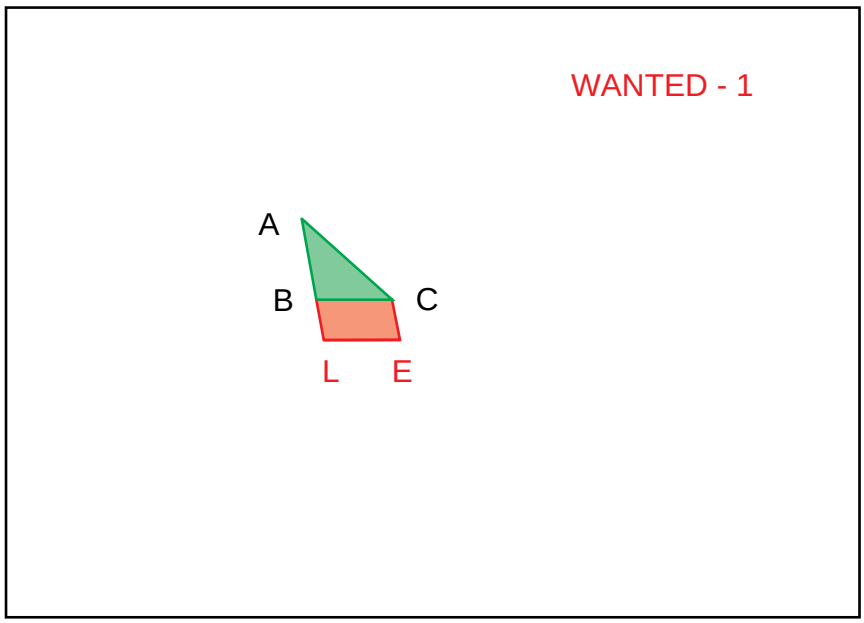


However, in stage 4, we may save a few steps by constructing the desired triangle KGH on top of the given triangle ABC, with G at B.

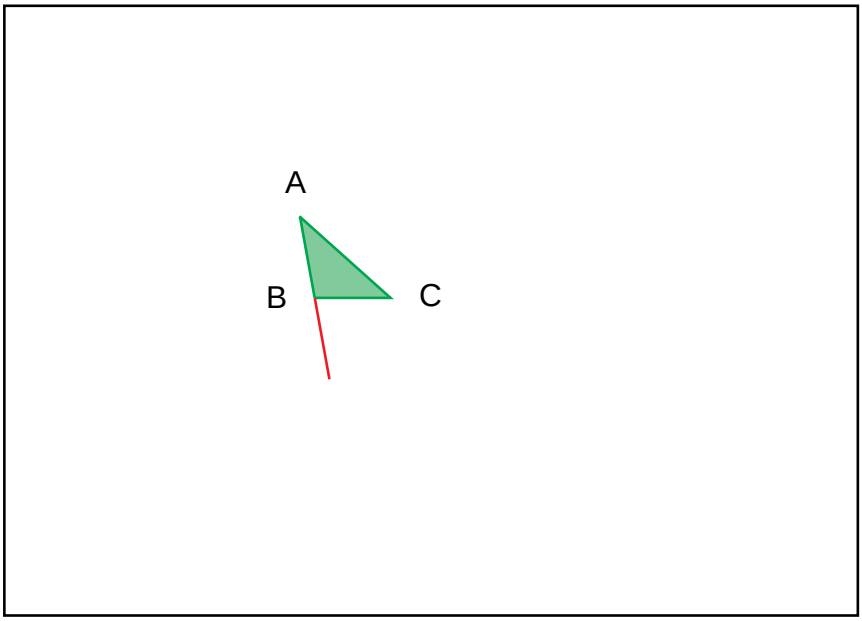


Stage 1.

VI.25:8. Let there be applied to BC the parallelogram BE equal to the triangle ABC [I.44],
We replace I.44 altogether with a shorter construction.

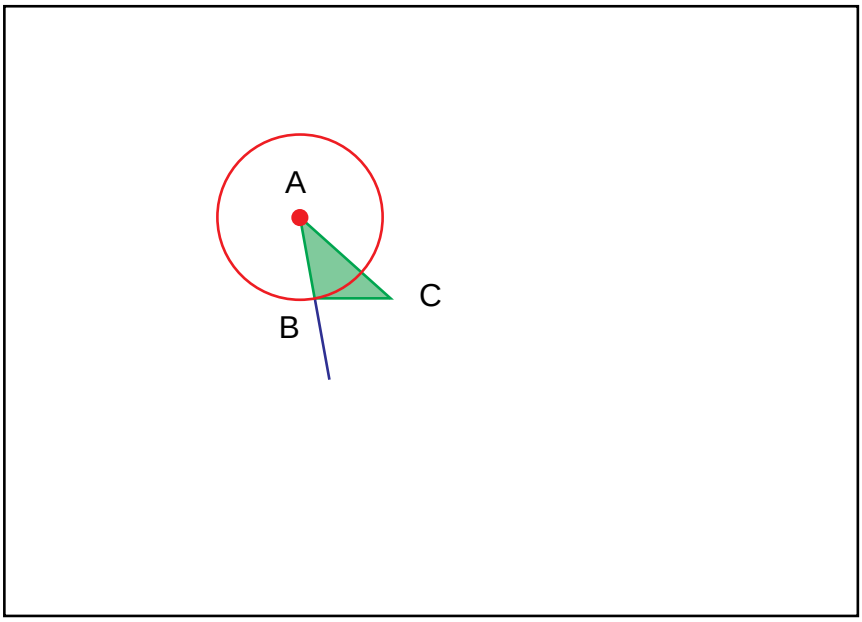


Extend the straight line AB downward.

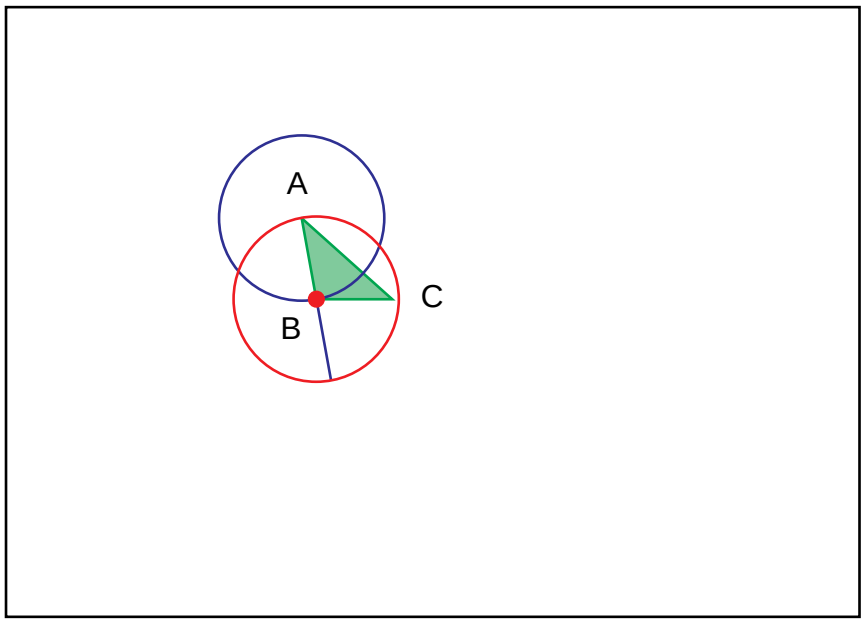


Bisect the straight line AB. [I.10]

GOSUB I.10, C#5B.
Describe the circle with centre A and distance AB.

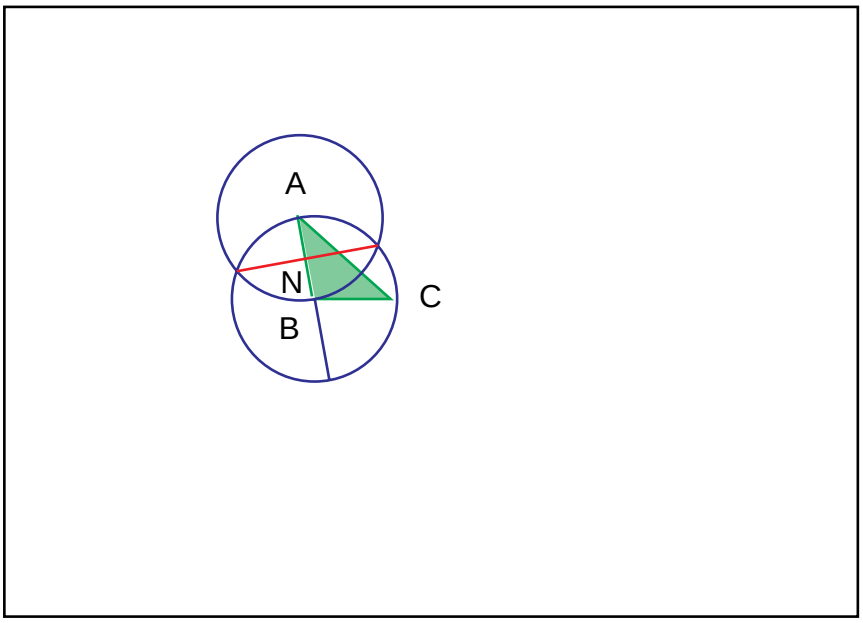


Describe the circle with centre B and distance BA.

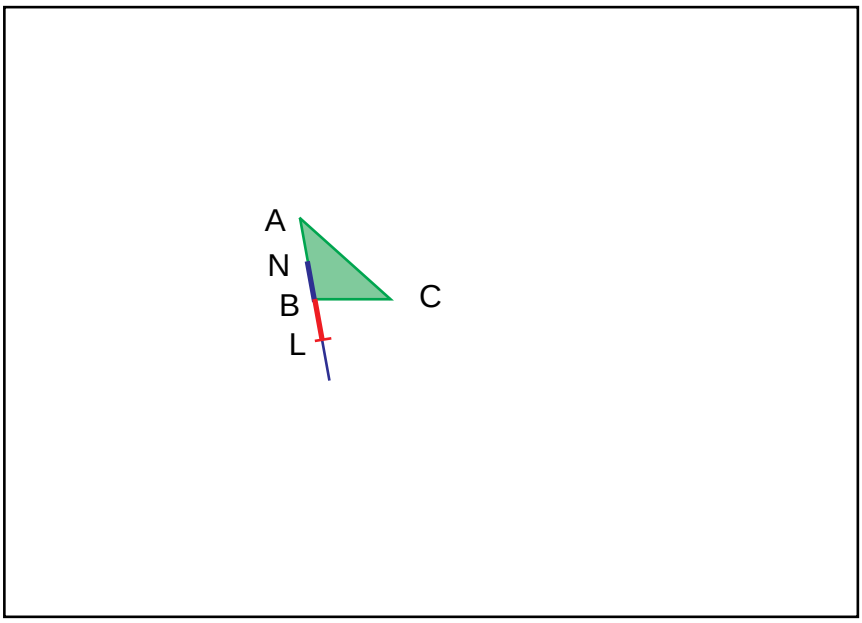


Connect the two crossing points of the circles to locate the mid-point N of the straight line AB.

RETURN to VI.25.
Cleanup.

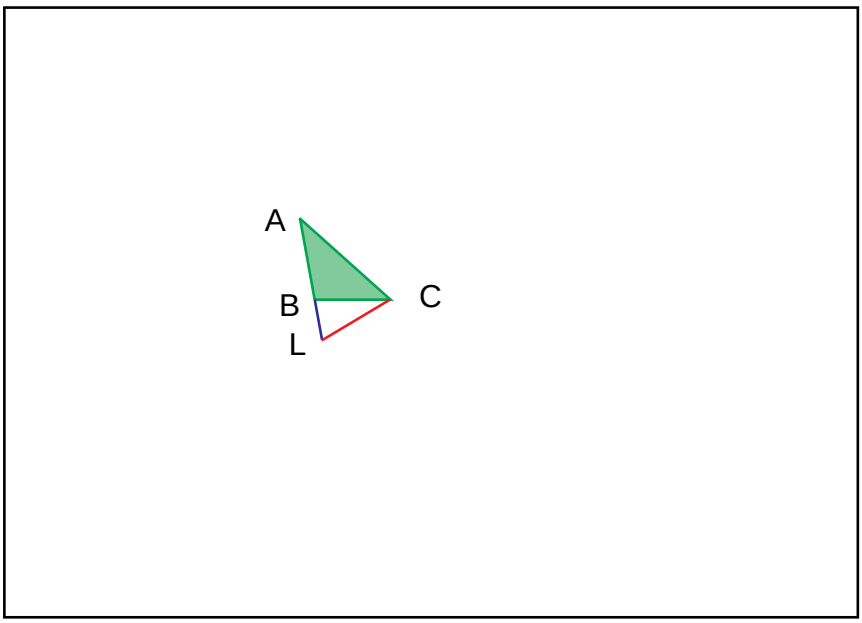


Measure the distance BN along the extension of AB to locate the point L. We now have two sides of the desired parallelogram for stage 1.

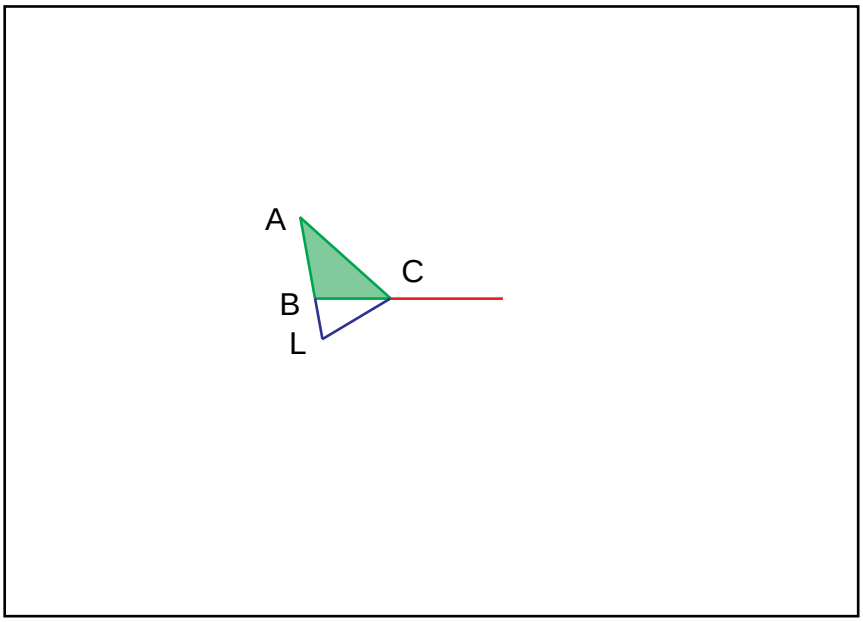


To complete the parallelogram, we might proceed directly, in classical fashion, by constructing the line through L parallel to BC, and then the line through C parallel to BL. The intersection of these two constructed lines would then locate the final point E of stage 1. But I.31, C#10, has ten steps, so altogether 20 steps would be required. Four triangles would be moved in this method, by the Proclus variation, C#8B. Instead, we will move just one.

Connect LC.

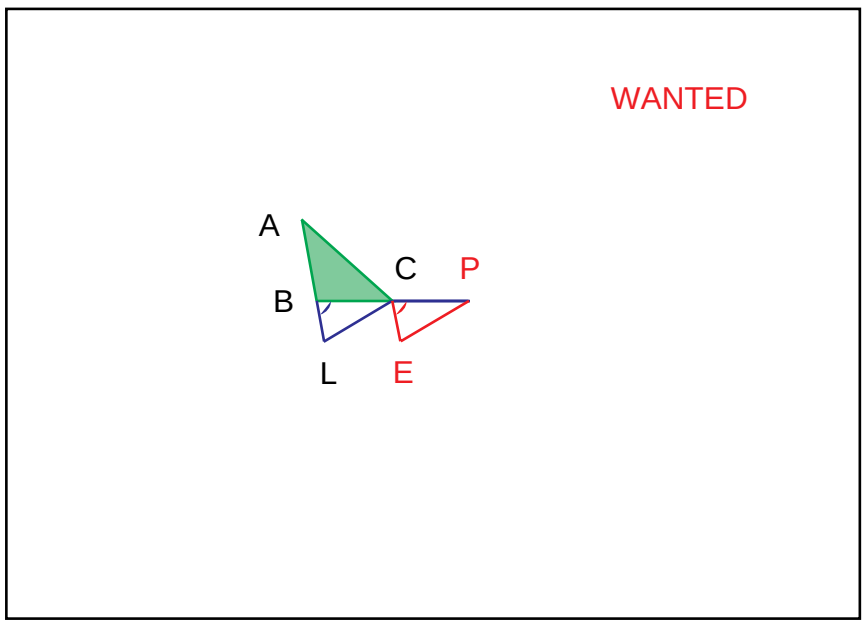


Extend the straight line BC.

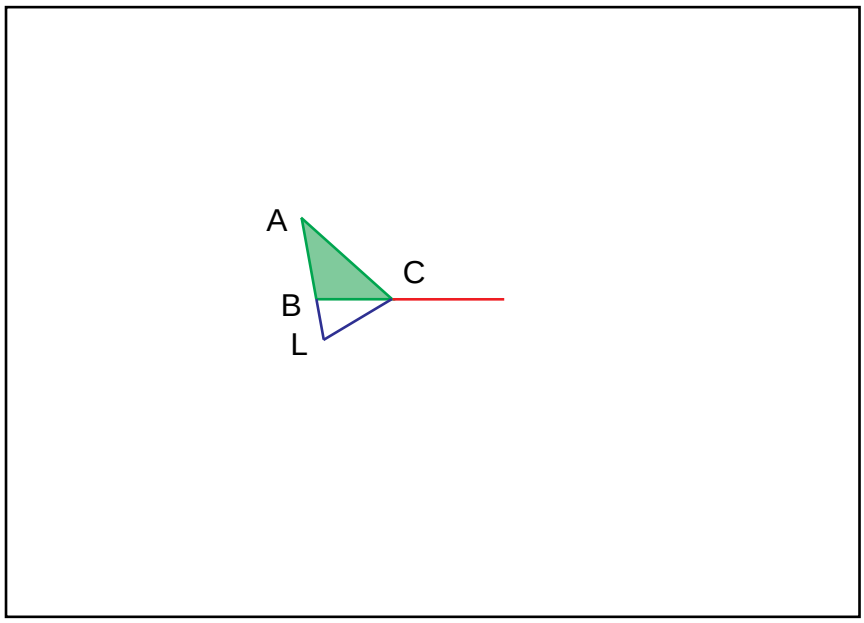


We will now move the triangle BCL so that the base BC moves onto the extension of BC, with B moving to C.

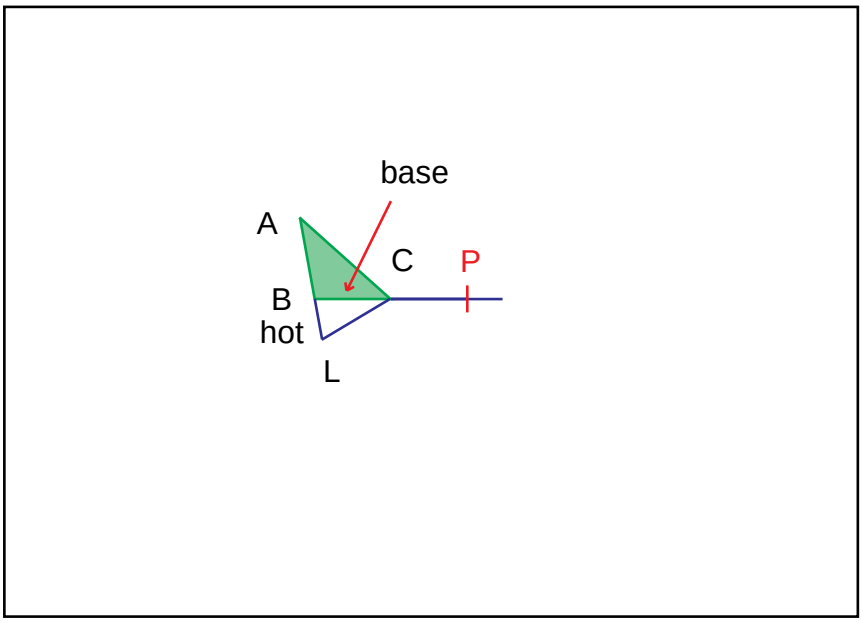
GOSUB C#8P.



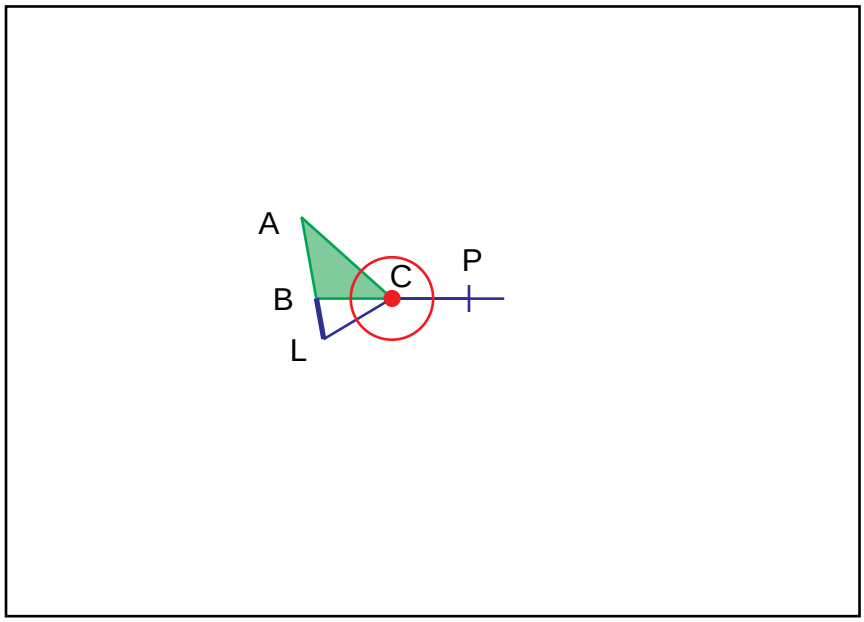
Extend the base BC.



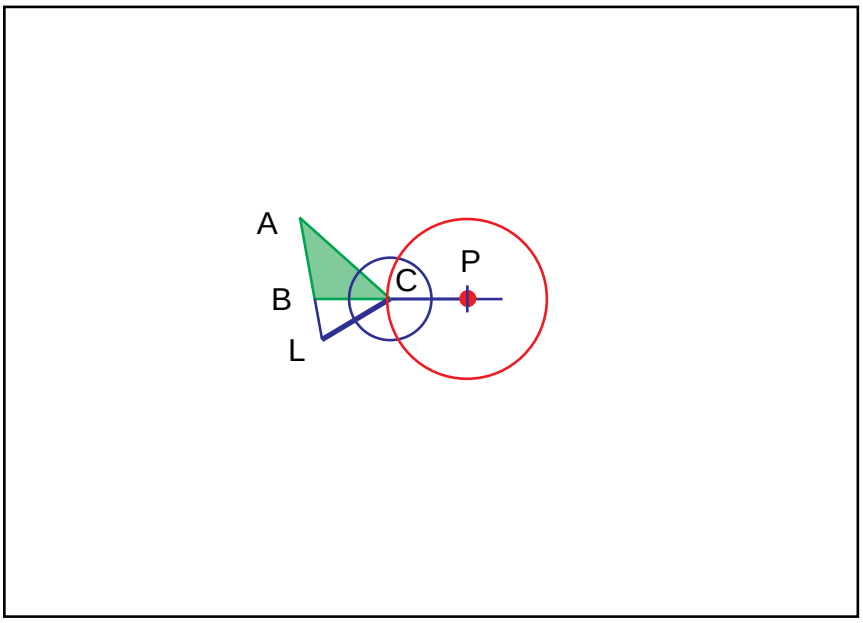
Move the base BC, so that B moves to C, to the moved base CP.



Swing the hot arm BL around the hot end C of the moved base CP.

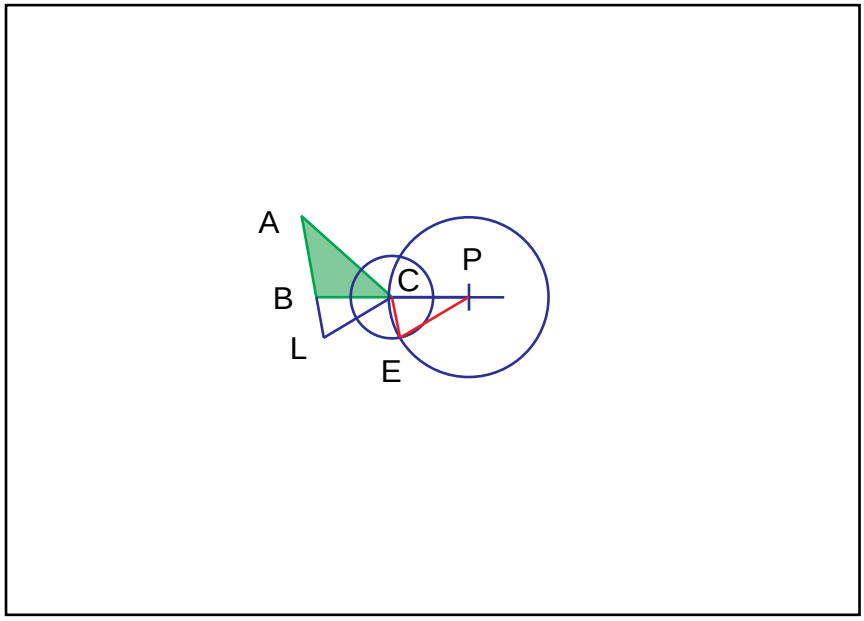


Swing the cold arm CL around the cold end P of the moved base CP.



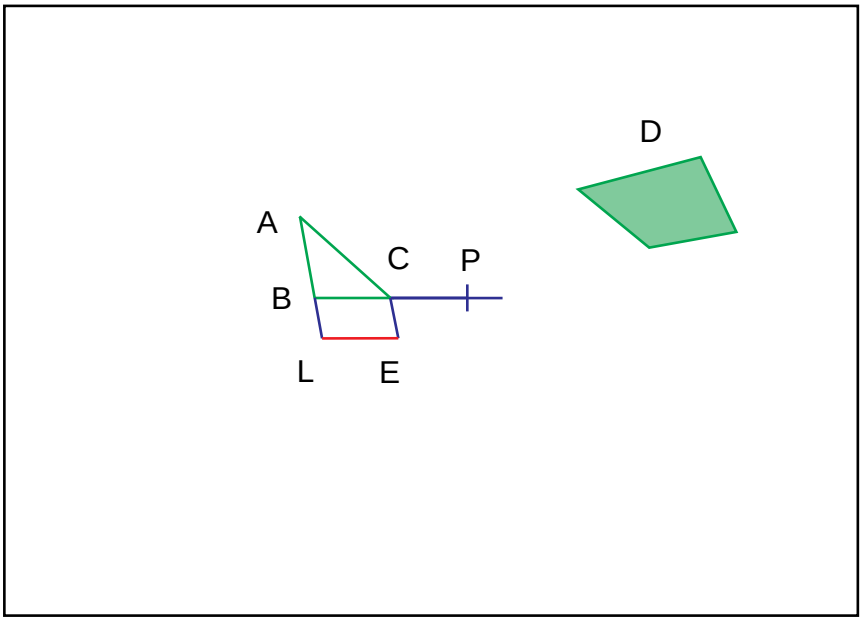
Connect the lower crossing point E to C and to P.

Now we have moved the triangle BCL to the triangle CPE. Actually this triangle is needed for the proof only; it is not needed for our construction. We will retain only the line EC. Similarly, we may now dispense with the line LC.



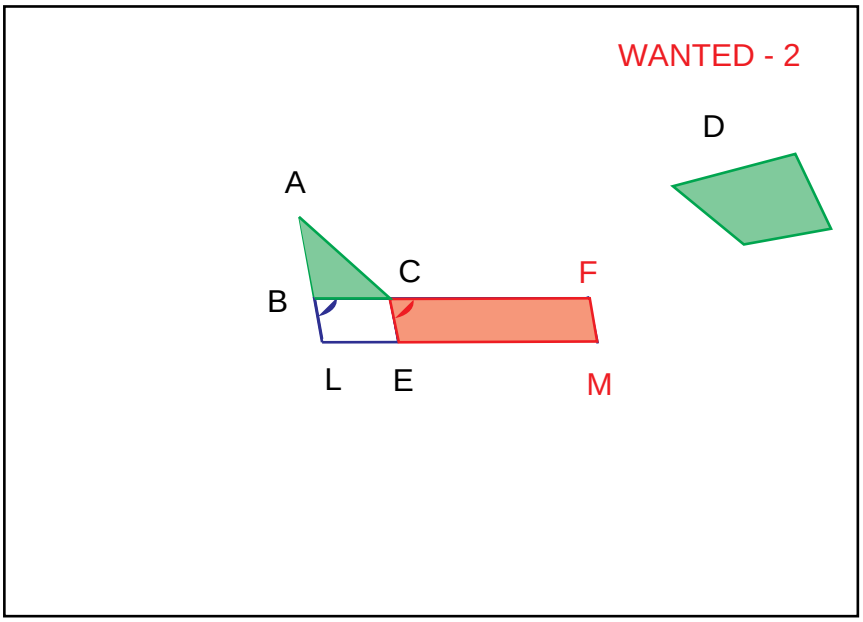
Cleanup.
Join LE.
RETURN to VI.25:8.

We now have attained Stage 1, with a drawing identical to Euclid's, except for our change in the angle CBL. We are ready for Stage 2, which will construct a parallelogram on CE, and in the angle PCE, which is equal to the angle CBL. At this point we may observe that Stage 1 is not entirely required! Only the first 5 steps are essential, as we could construct our Stage 2 parallelogram on BL, which is equal to CE, and in the angle CBL. However, we will follow Euclid here, and just note for the record that 8 of our steps in Stage 1 are optional.



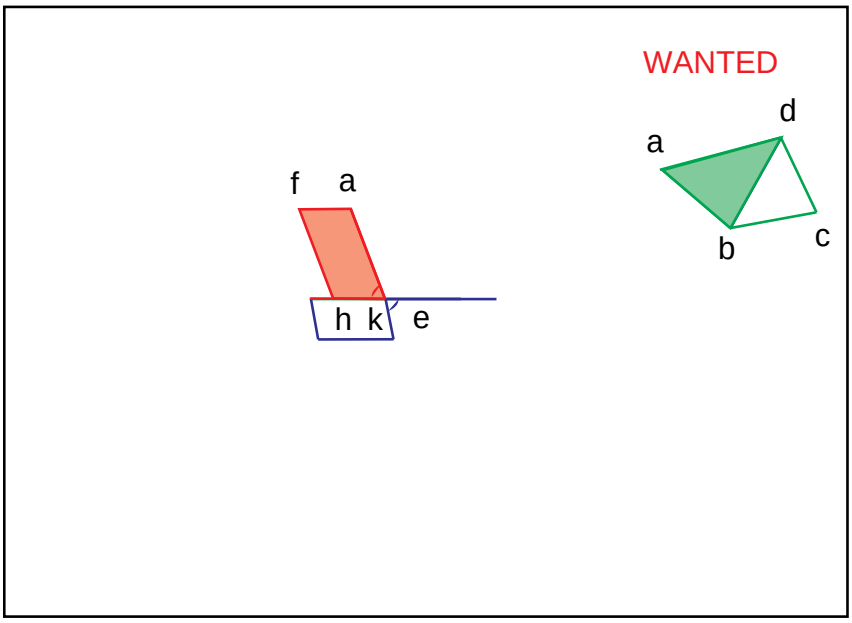
Stage 2.
VI.25: and to CE the parallelogram CM equal to D in the angle FCE which is equalk to the angle CBL. [I.45]

In this stage we will hide, temporarily, the given triangle ABC, and label the verrtices of D with the letters a, b, c, d. The strategy of I.45 (C#13) is to divide D into triangles, then move the triangles one at a time, using I.44 (C#12) repeatedly. In turn, I.44 (The Mt. Everest of Book I) uses I.42 (C#11, which we have just called in Stage 1) to move the triangle to a parallelogram in a strategic position, and then, the Clever Flip. We begin with the first of the two triangles.

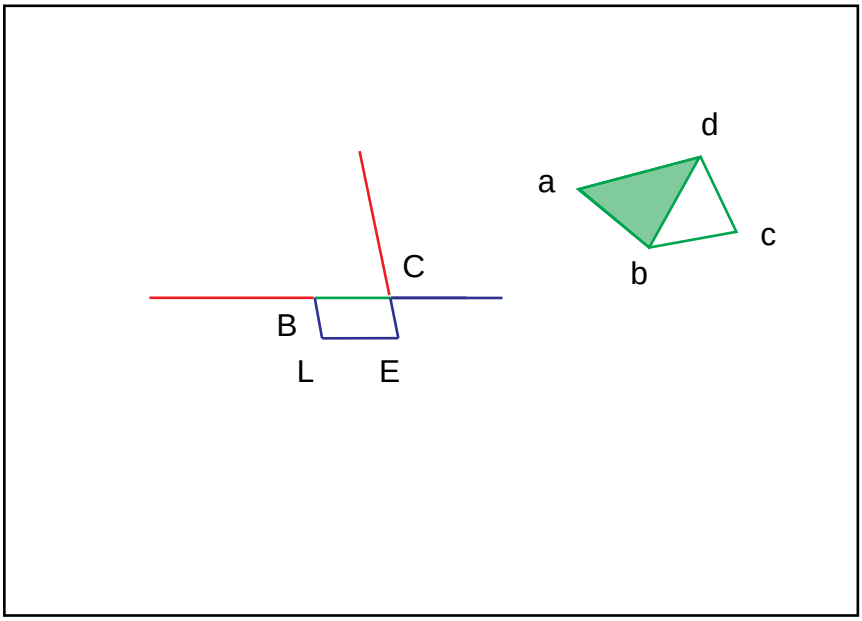


GOSUB I.45.
I.45:7. Let db be joined, and let the parallelogram fh be constructed equal to the triangle abd, and in the angle hkf, which is equal to e; [I.42]

GOSUB I.42. As we want the parallelogram constructed in the position shown above, we begin by moving the triangle abd to the angle e, rather than moving the angle to the triangle, as Euclid does in I.42.

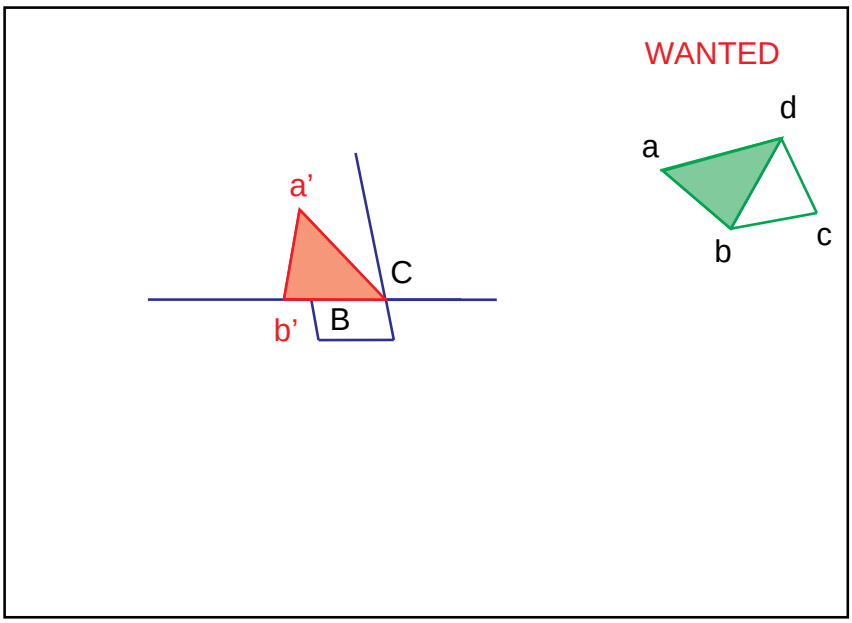


Extend CE upward and CB left-ward.

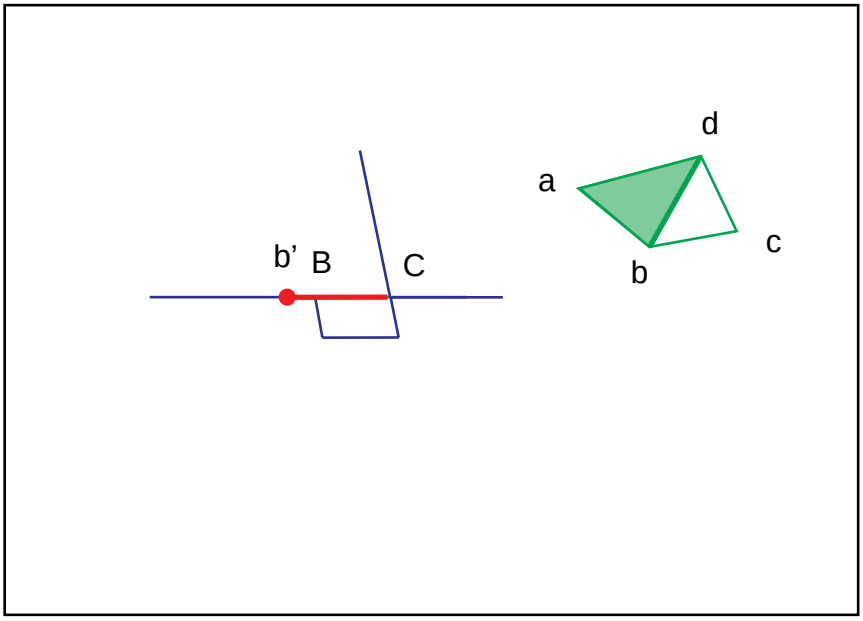


Move the triangle abd to the line BC extended so that d moves to C. ([I.22])

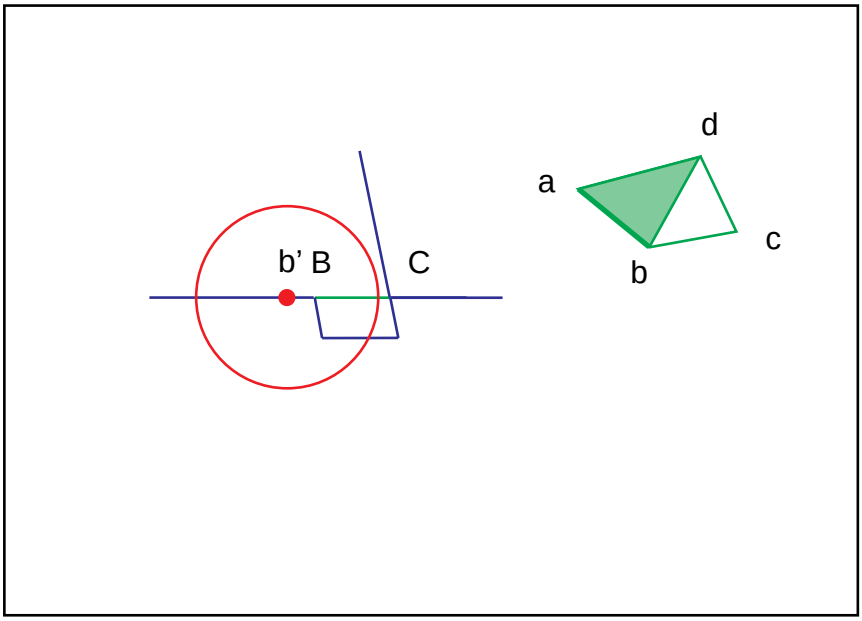
GOSUB I.22, C#8.



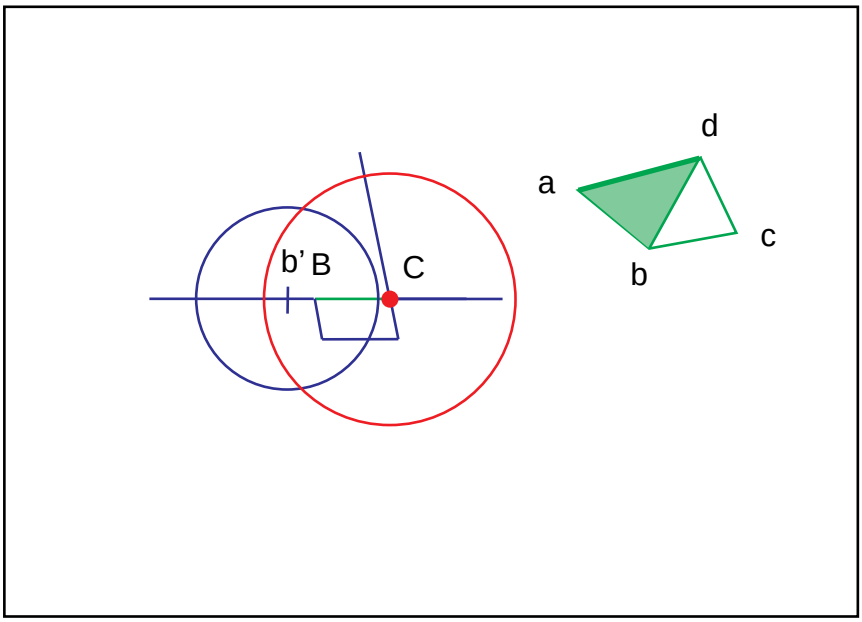
Move the side bd to the line BC extended, b moving to C, and b to b'.



Swing the arm ab around b'.

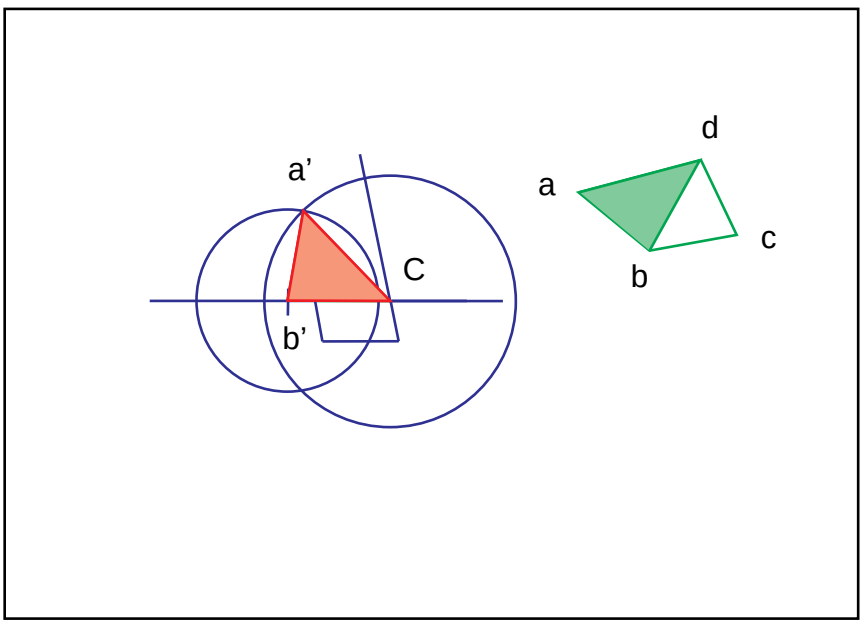


Swing the arm ad around C.



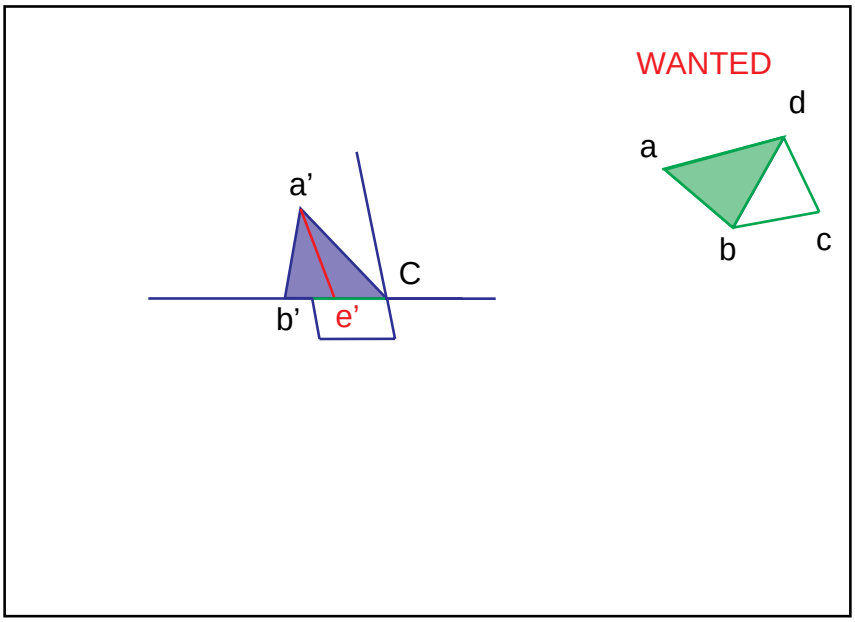
Connect the upper crossing point a’ to b’ and C.

Cleanup.
RETURN to I.42.



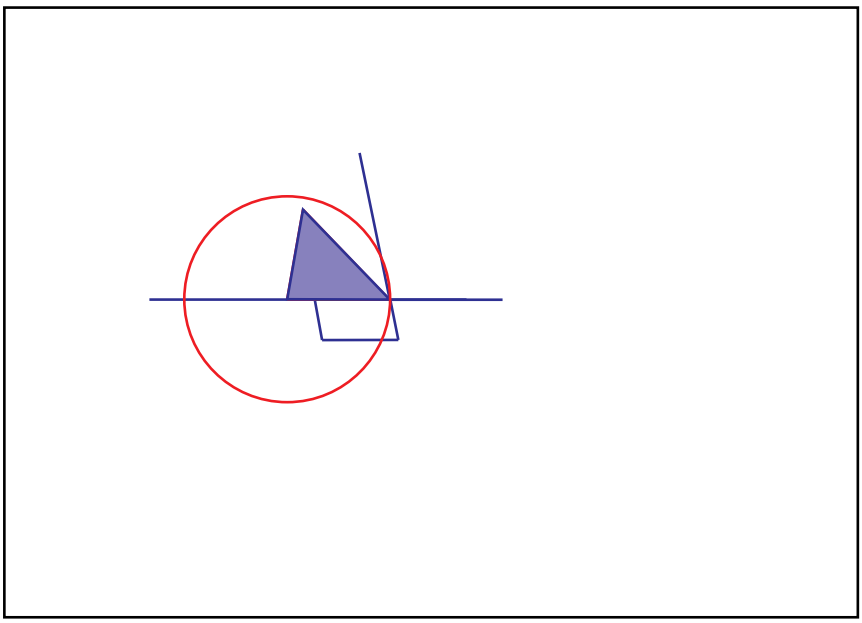
Next we must transform the moved triangle into a parallelo-gram of the same size, by cutting it in half. We rejoin I.42 at line 8. I.42:8. Let b’C be bisected at e’,

and let a’e’ be joined; ([I.10])

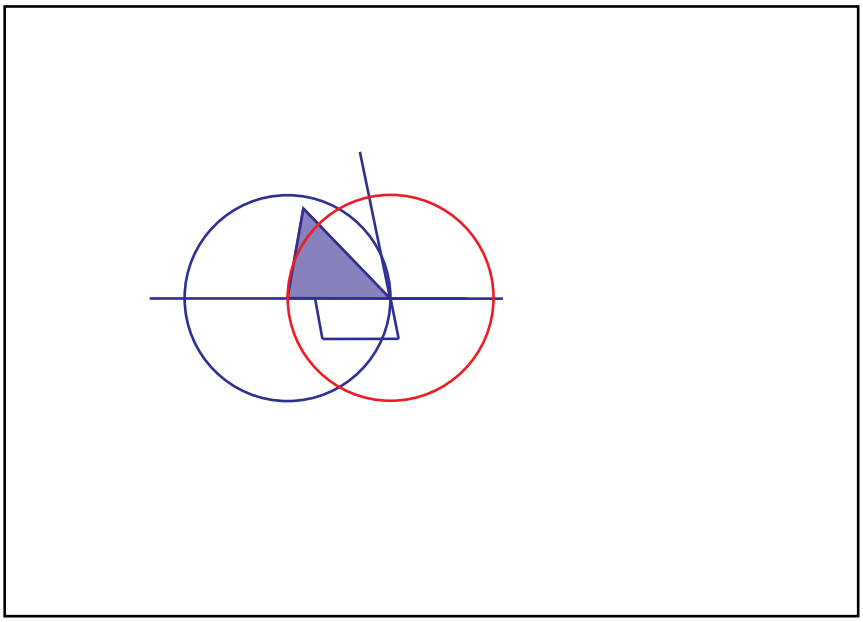


GOSUB I.10, C#5.

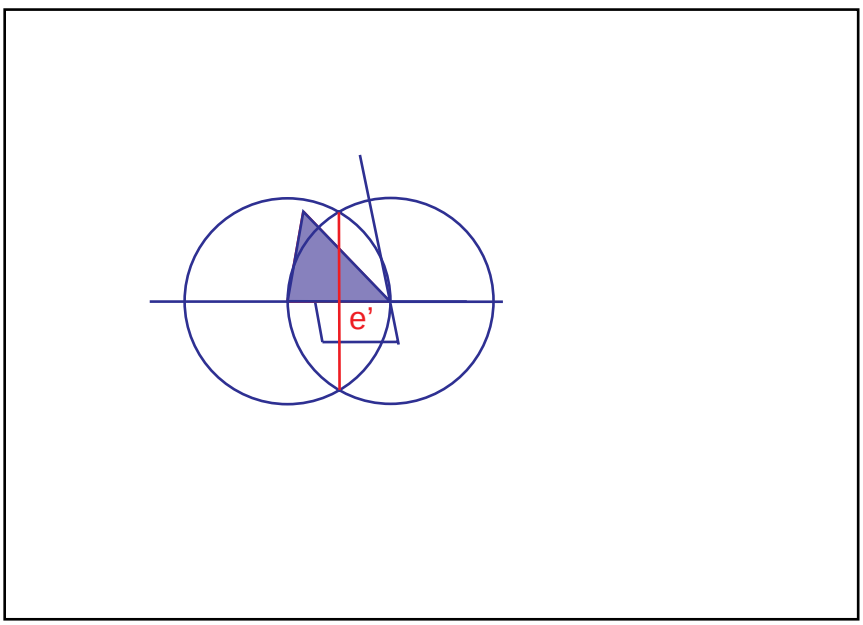
Swing b’C around b’.



Swing Cb’ around C.



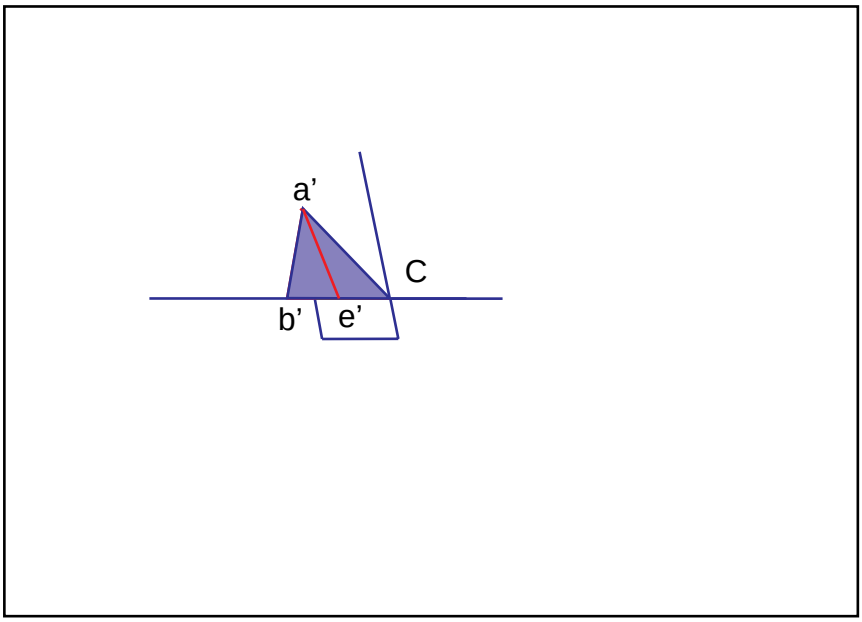
Connect the two crossing points to locate the point e’ on b’C.



Cleanup.
RETURN to I.42:8.

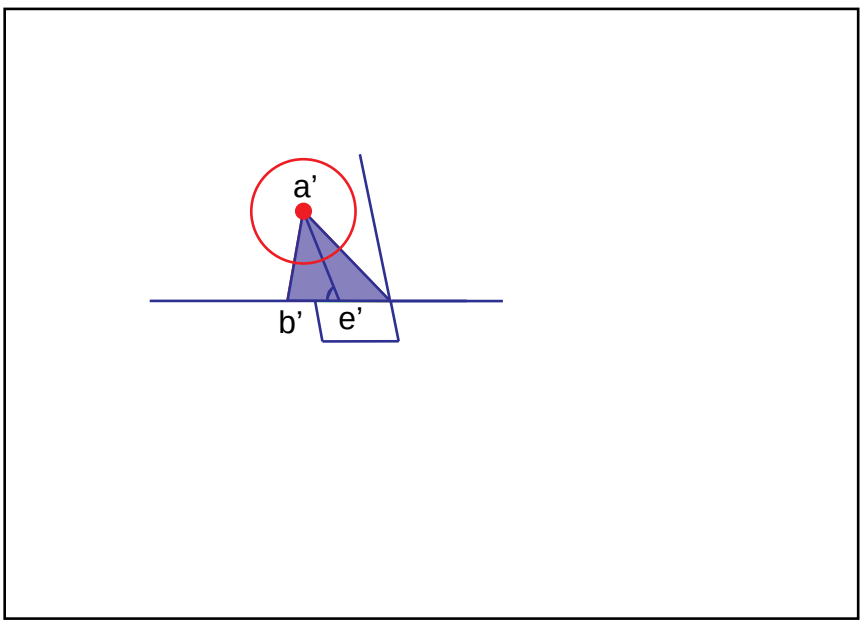
I.42:9. and let a’e’ be joined;

RETURN to I.45.

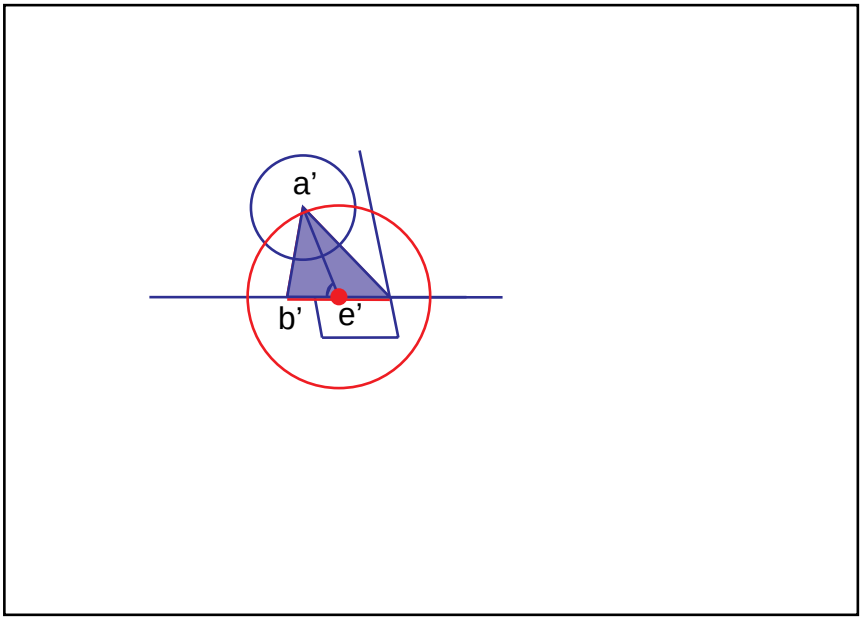


Next we must move the triangle a’b’e’ such that the base a’e’ rotates about its midpoint, and the angle a’e’b’ moves to e’a’. We are still at the beginning of Stage 2, in our variation of I.45.

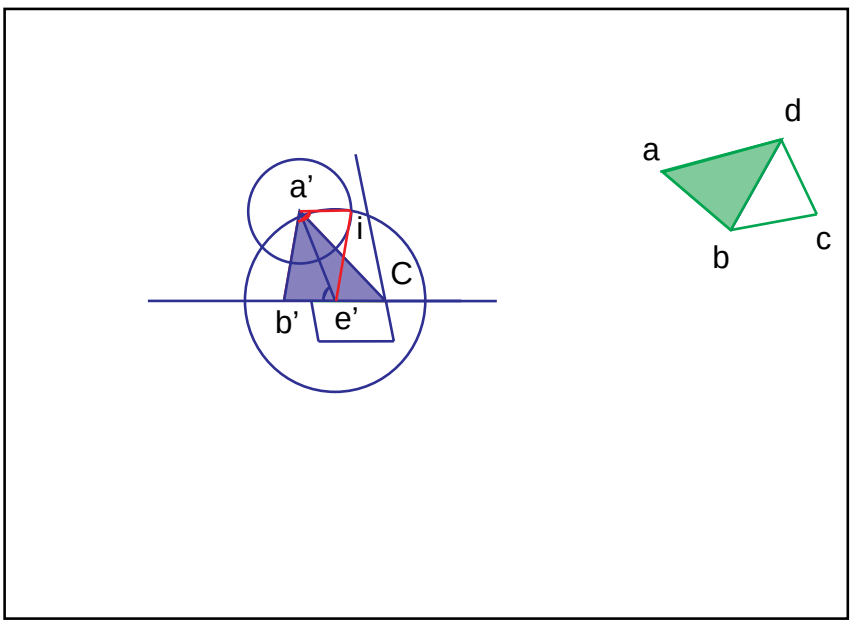
Swing the hot arm b’e’ around the hot end a’ of the moved base a’e’.



Swing the cold arm a’b’ around the cold end e’ of the moved base a’e’.

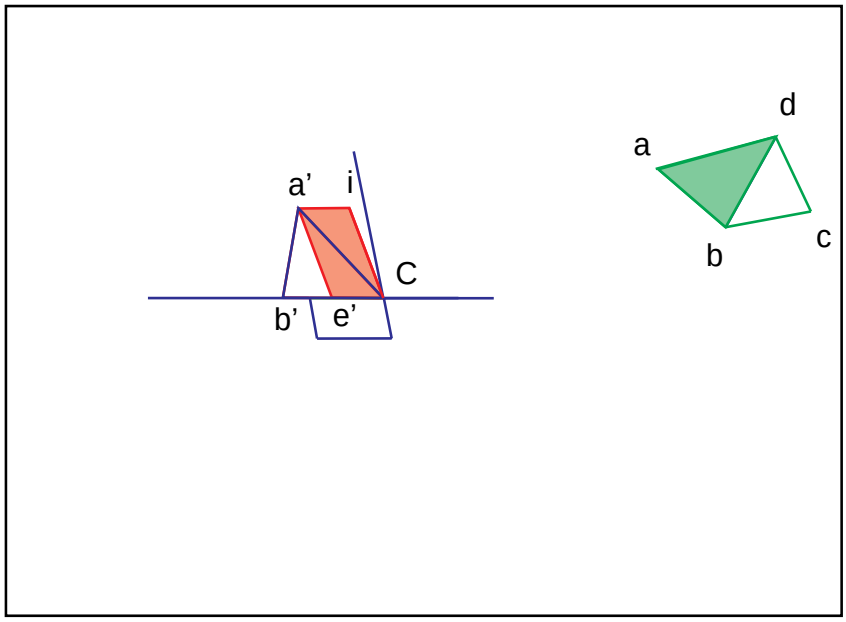


Connect the upper crossing point i’ to the two ends of the (moved) base, ci’a’, i’e’.



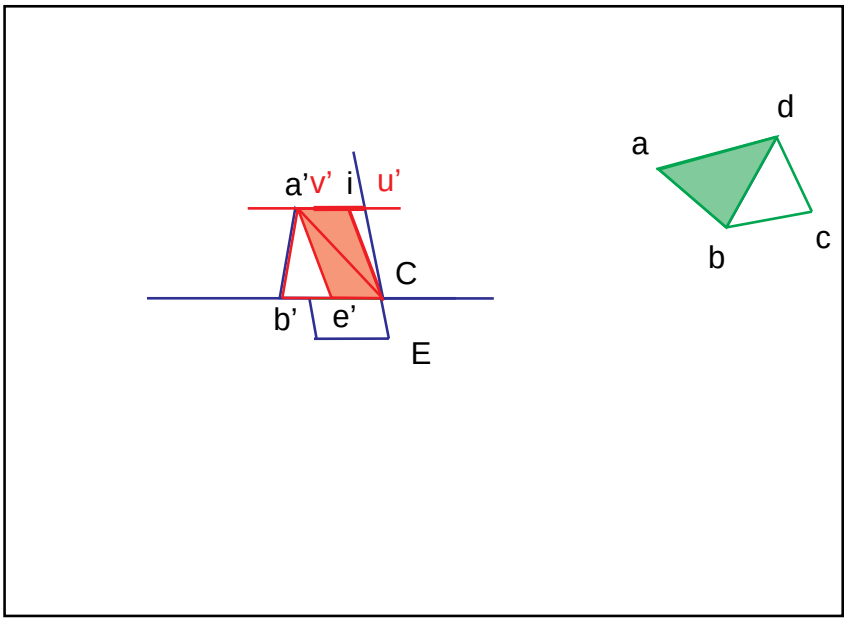
The angle i’a’e’ is equal to the angle a’e’b’. The parallelogram a’b’e’i’ is equal to the triangle a’b’C, and to the triangle abd.

Cleaunup.
Connect i'C.
The parallelogram a'e'Ci' is also equal to the triangle a'b'C, and to the triangle abd.

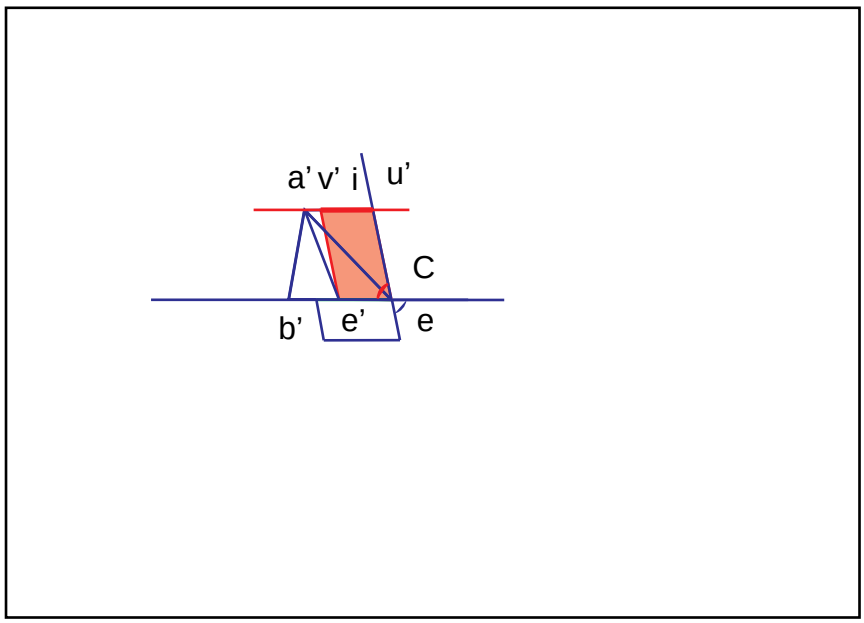


We next want to move the parallelogram a'e'Ci' so that its side Ci' is aligned with the line CE.

Extend the line a'i', let u' denote the point in which it meets the extension of CE, and locate v' on it so that u'v' is equal to Ce'.



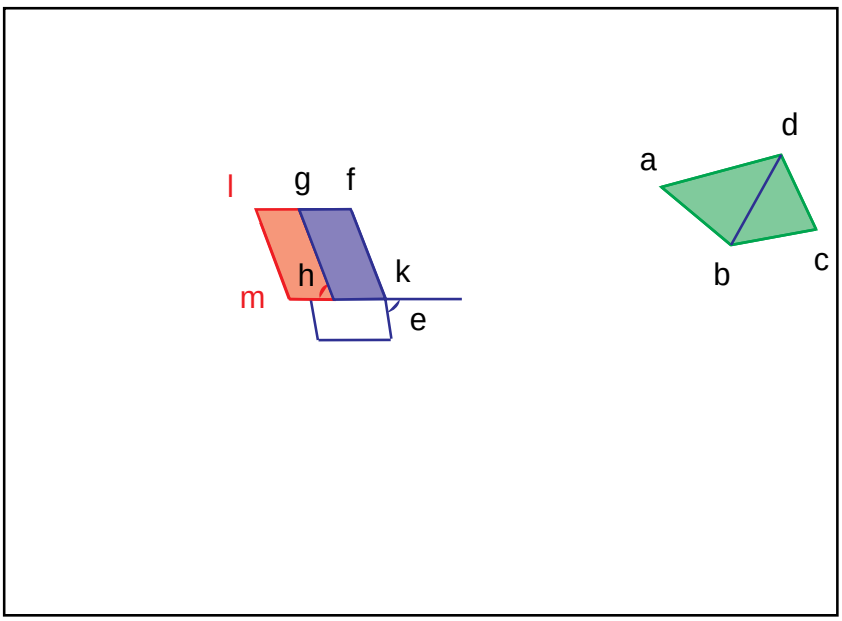
Connect v'e'. The parallelogram u'v'e'C is equal to the triangle a'b'C, and the angle b'Cu' or hkf, which is equal to e.



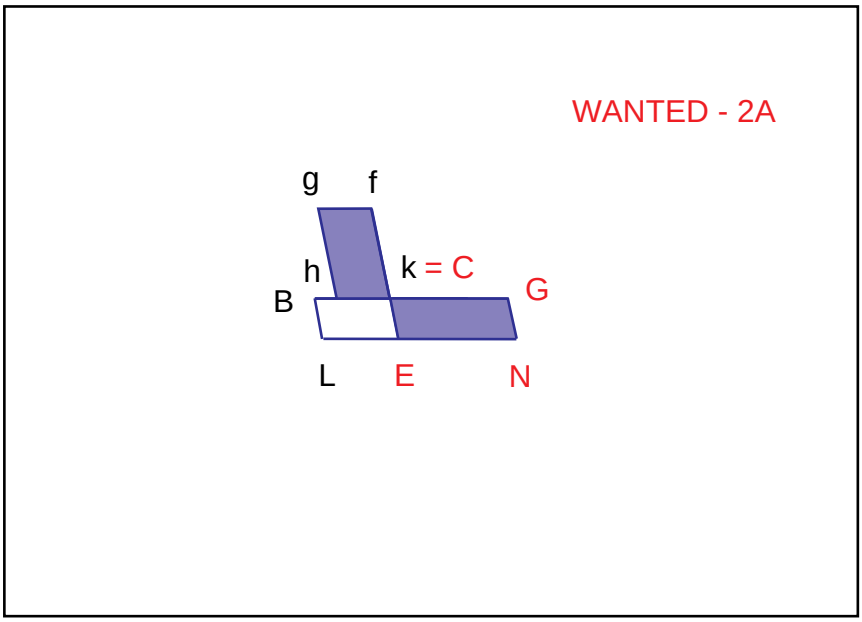
This completes the call to I.42.

RETURN to I.45:7.
Cleanup. Relabel.

I.45:10. let the parallelogram gm equal to the triangle dbc be applied to the straight line gh, in the angle ghm which is equal to e. [I.44]



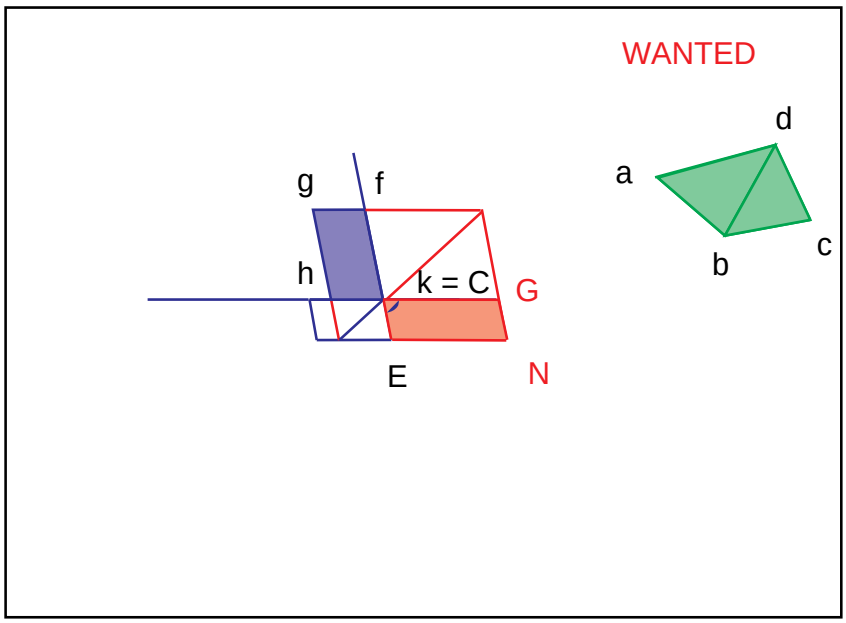
But, for reasons of economy, we want to apply the oparallelogram fghk tio the line CE first.



This requires the Clever Flip, the main step of I.44, which is justified in I.43.

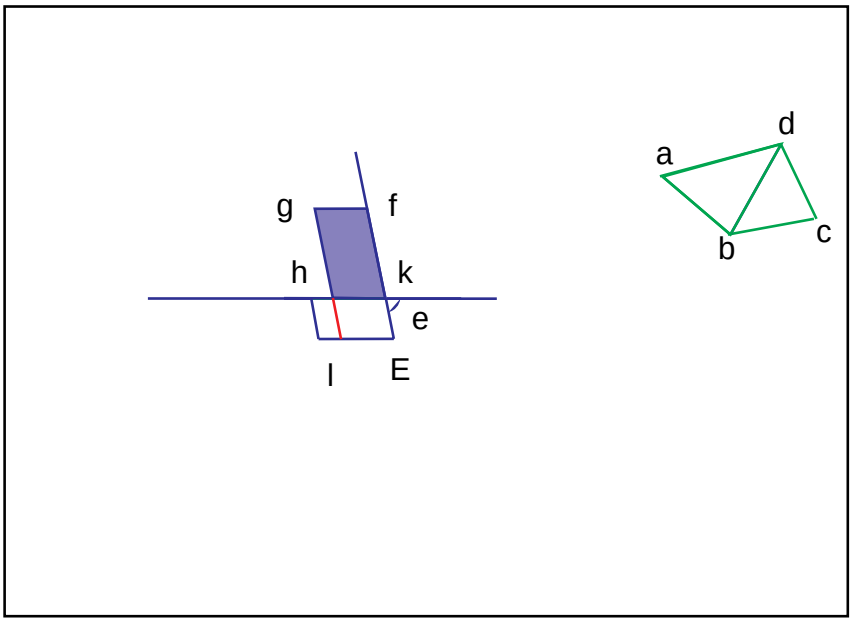
I.45:10 (modified). let the parallelogram fh equal to the triangle abd be applied to the straight line CE, in the angle e. [I.44]

Later, in Stage 2B, the triangle bcd will be applied directly to the line GN, completing Stage 2..

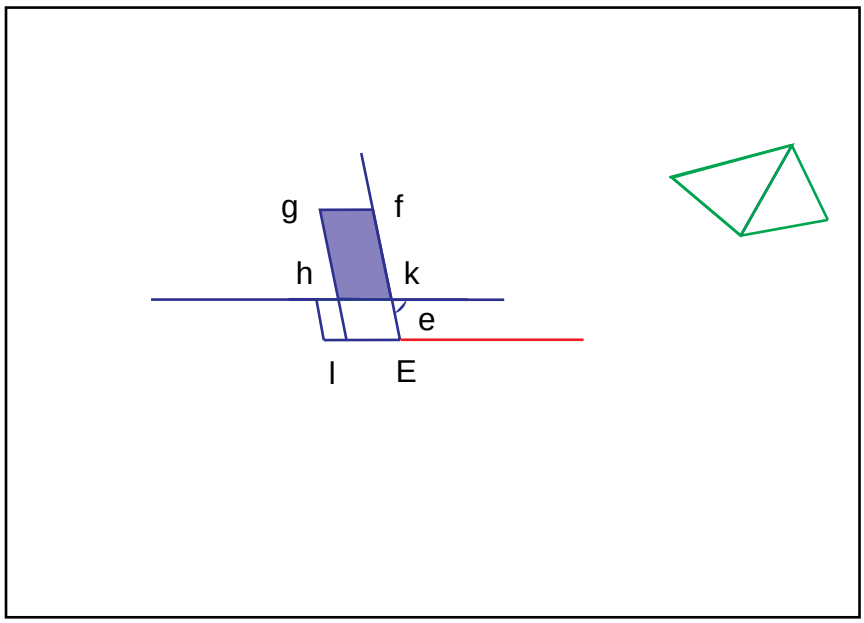


GOSUB I.44.
Keep labels.

I.44:10. let gh be drawn through to l,

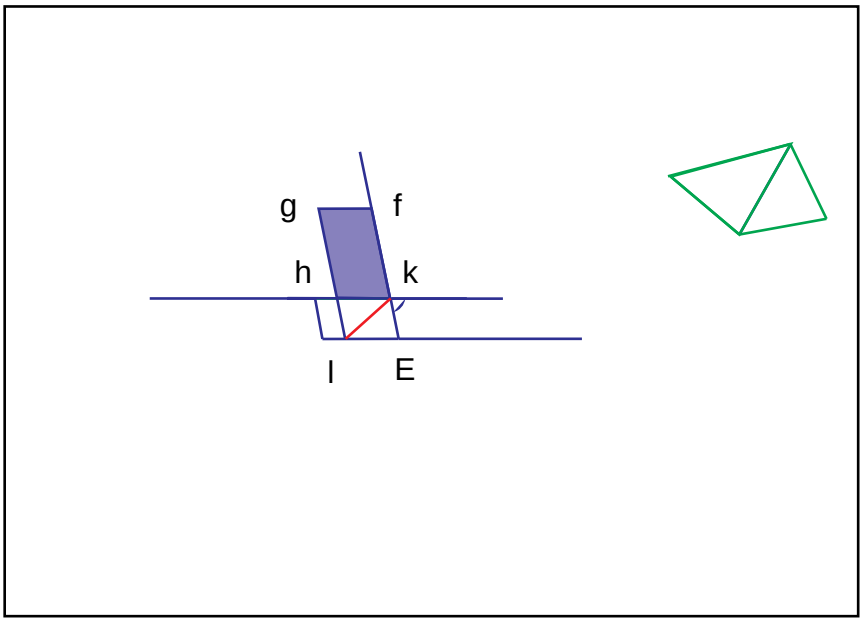


I.44:11. and let El be drawn through E parallel to either kh or fg. [I.31]



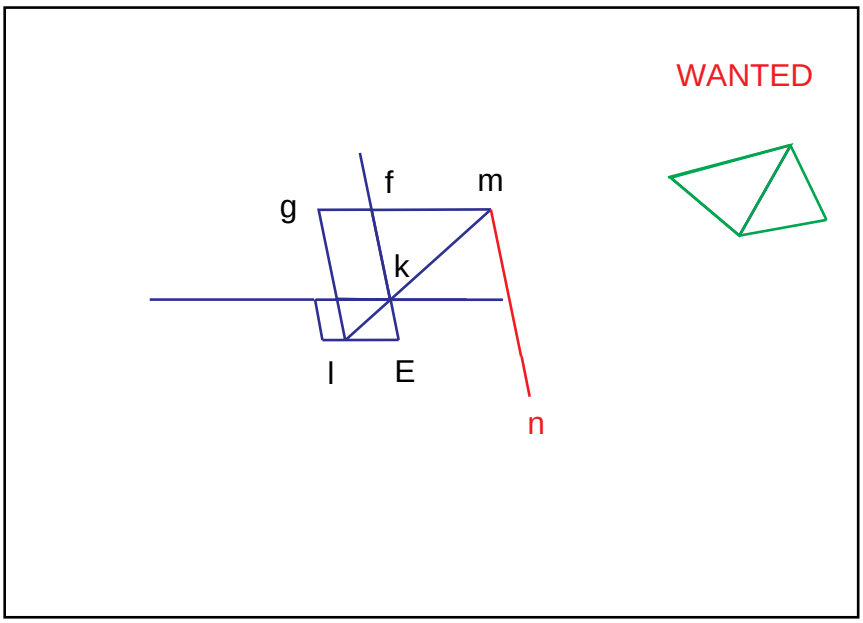
We do not need I.37 here, as we have already constructed lE parallel to kh and fg. Now we simply extend lE.

I.44:13. Let lk be joined.

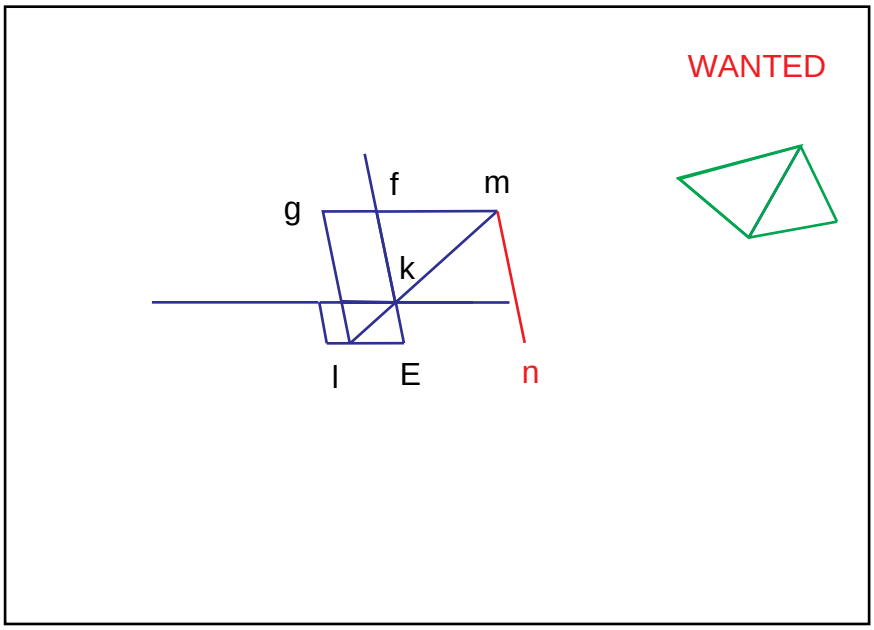


I.44:21. lk, gf, when produced, will meet.

Let them be produced and meet at m;

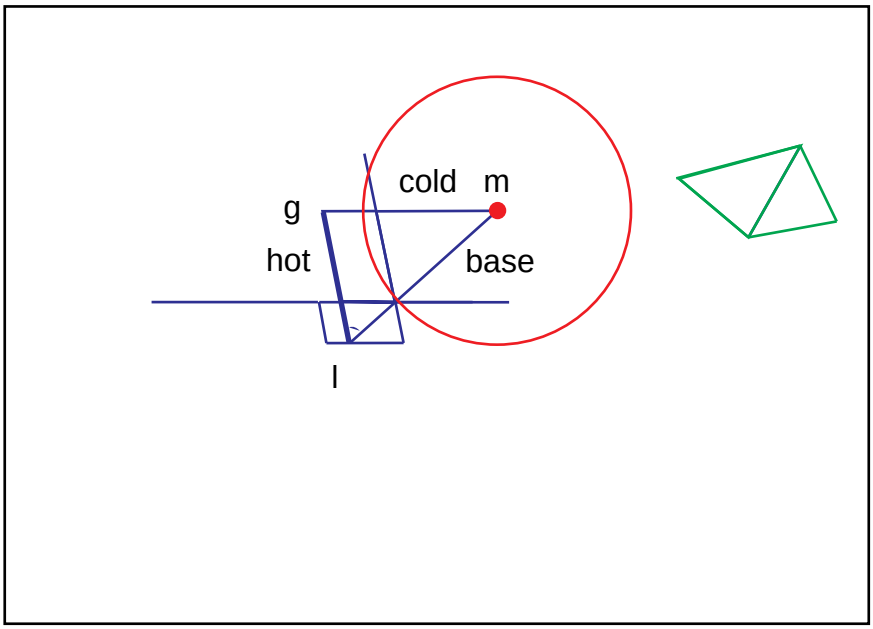


I.44:22. Through the point m let mn be drawn parallel to either fe or gl, [I.37]

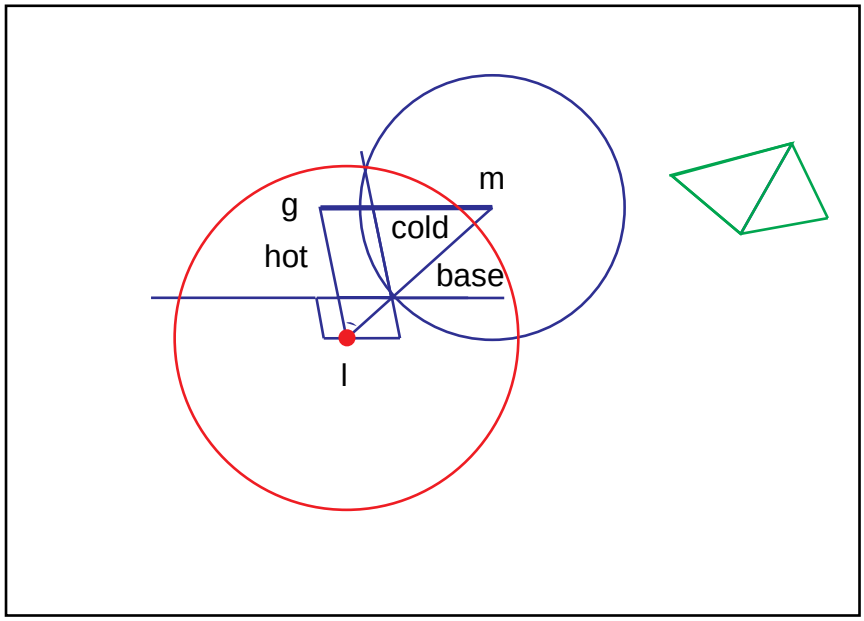


As usual, I.31 (C#10) calls I.23 (C#9), which in turn calls I.22 (C#8) but needs the Proclus Variation, C#8P. We will go to our alternative, C#10B. This rotates the triangle glm around the midpoint of its base, lm. This gives us, for free, the line ln parallel to gm, which is needed in the very next step of I.44.

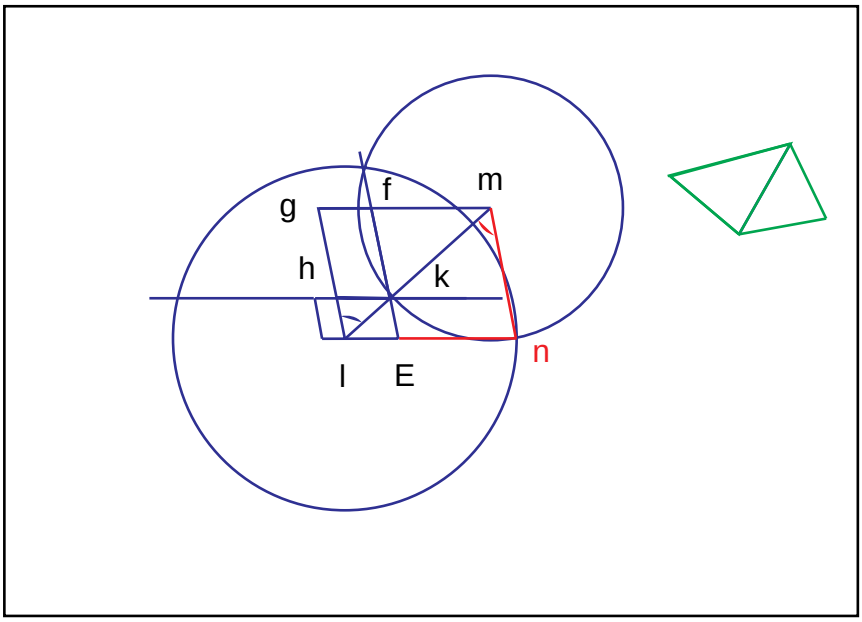
Swing the hot arm gl around the hot end m of the moved base ml.



Swing the cold arm gm around the cold end l of the moved base ml.



Connect the lower crossing point n to the two ends of the moved base.



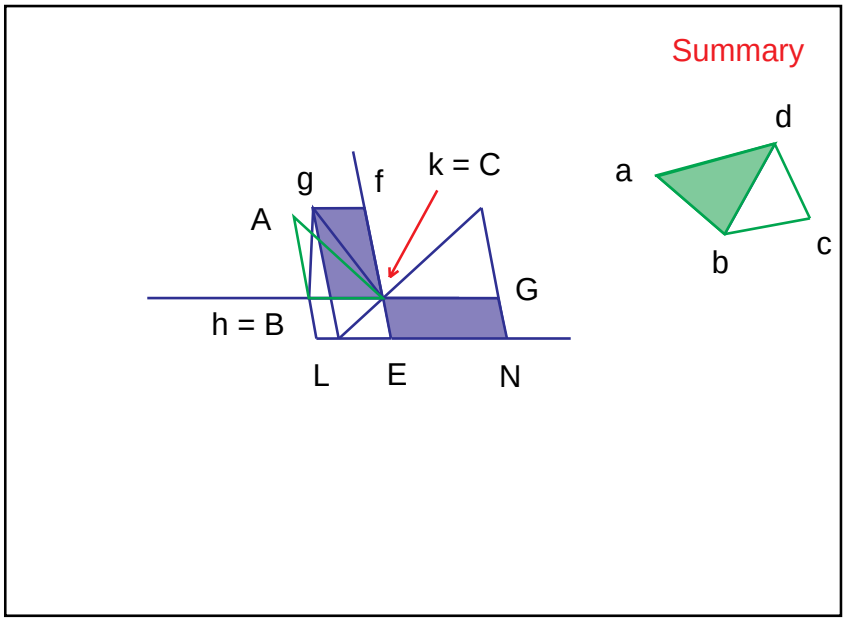
Cleanup.
RETURN to I.44:22. Relabel.

We have achieved Stage 2A.
RETURN to I.45.
Summary.

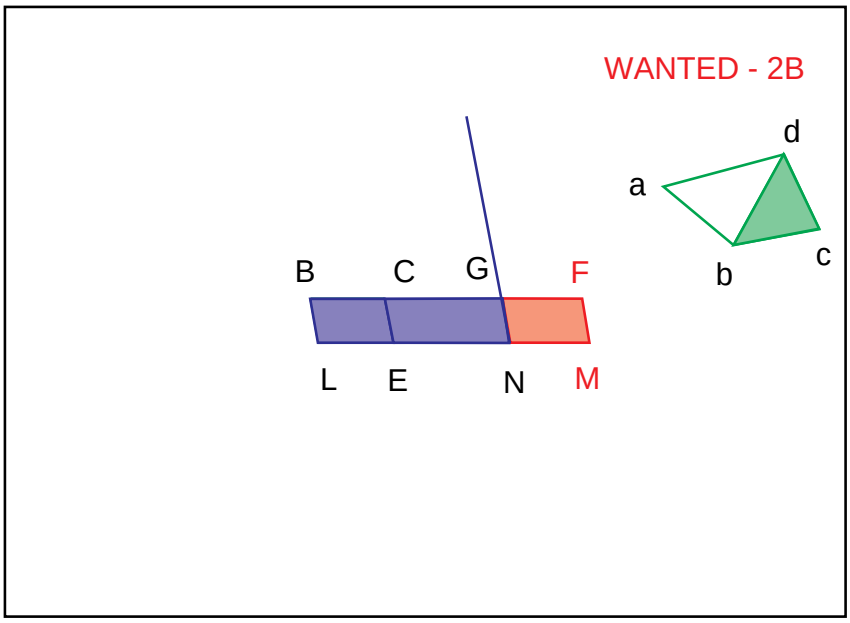
The triangle abd, the triangle ghk, the parallelogram gk, and the parallelogram EG: all are equal (Stage 2A).

The triangle ABC, and the parallelogram BE, are equal (Stage 1).

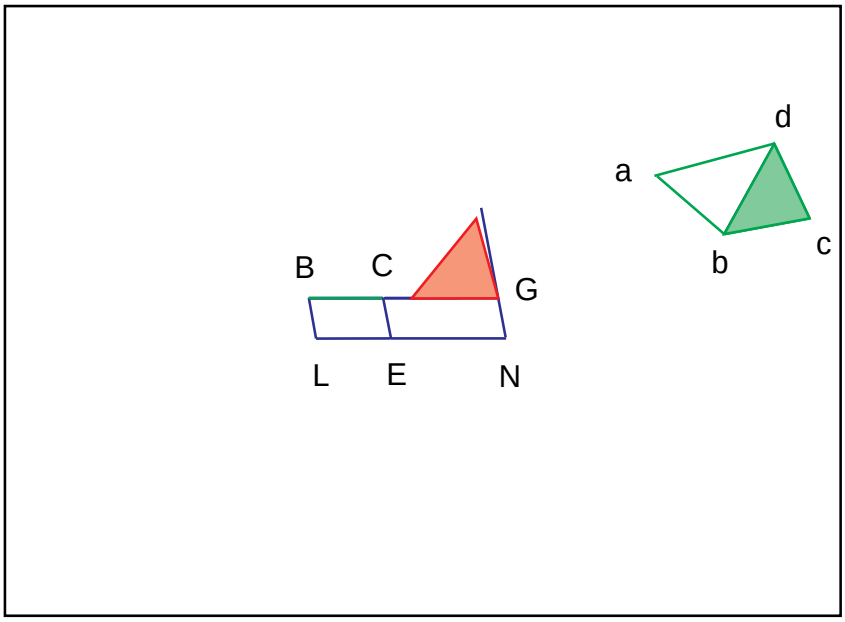
Cleanup for stage 2B.



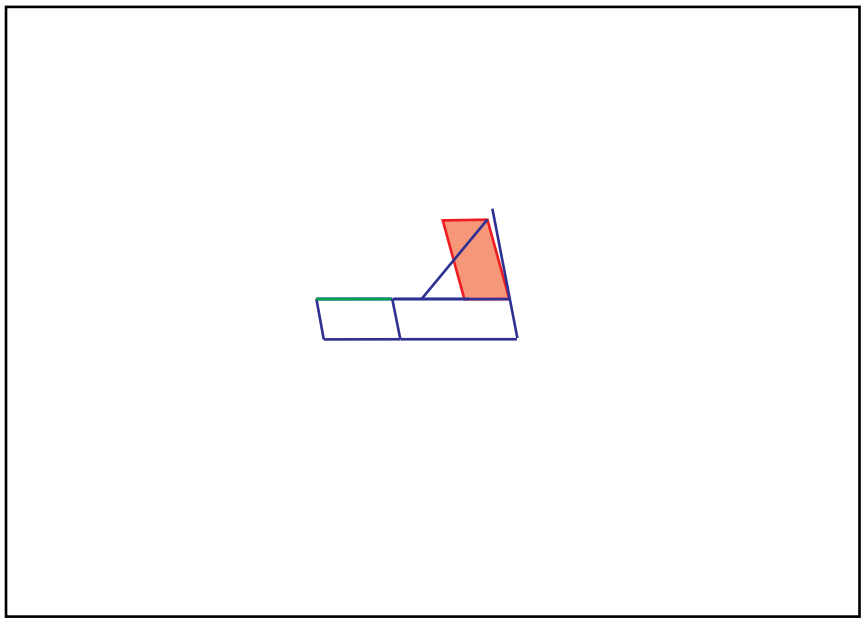
Stage 2B.
Let the parallelogram GM equal to the triangle bcd be applied to the straight line GN.



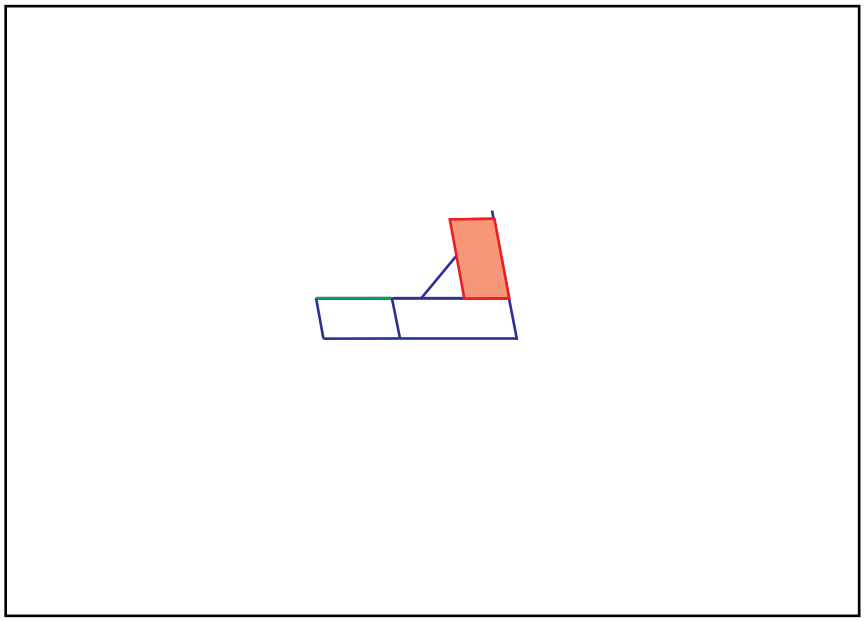
We now repeat the steps of Stage 2A, steps #15 to 37. We will give only an outline, like I.44.
Move triangle bcd onto the line CG. (4 steps, like 15, 16, 17, and 18).



Transform the triangle into a parallelogram. (Eight steps, like steps 19-26.)

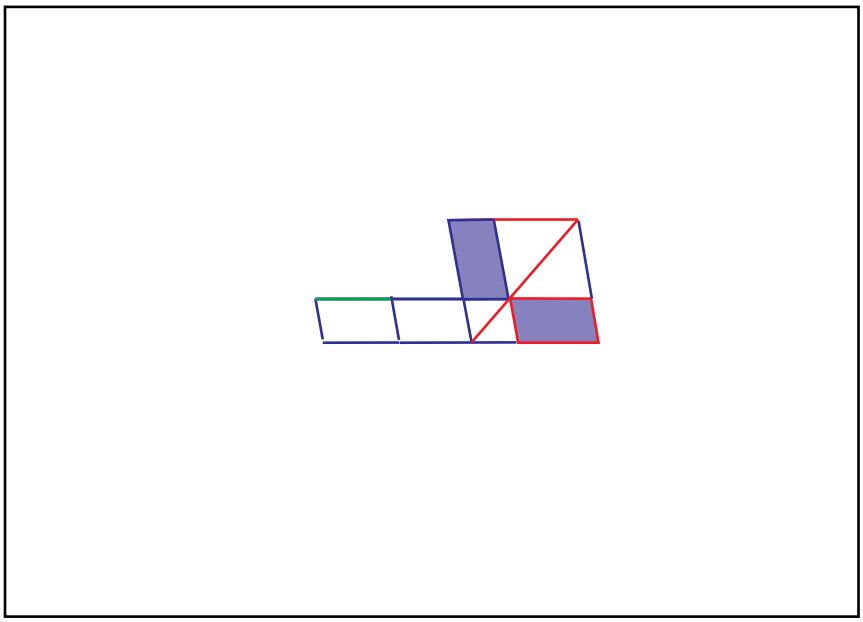


Reshape the parallelogram to the desired slope.
(Two steps, like steps 27, 28.)



The Clever Flip.

(Seven steps, like steps 29 - 35.)

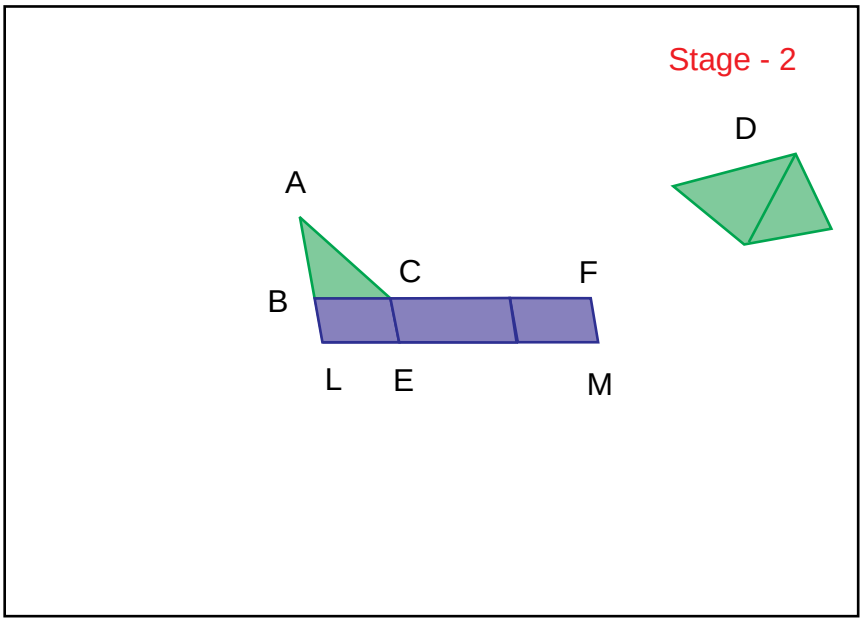


Cleanup.

We have now completed Stage 2.

We have BE equal to ABC, and CM equal to D.

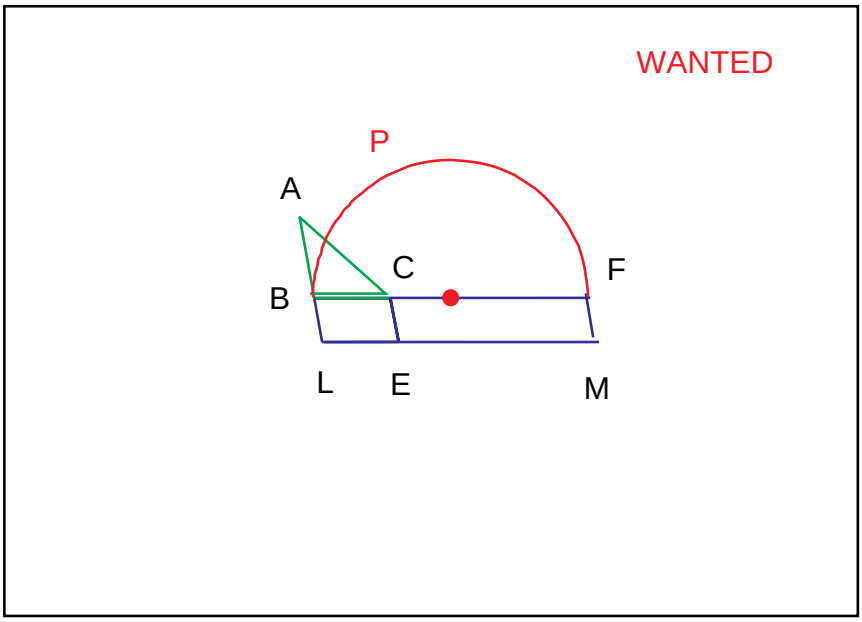
RETURN to VI.25 at line 9.



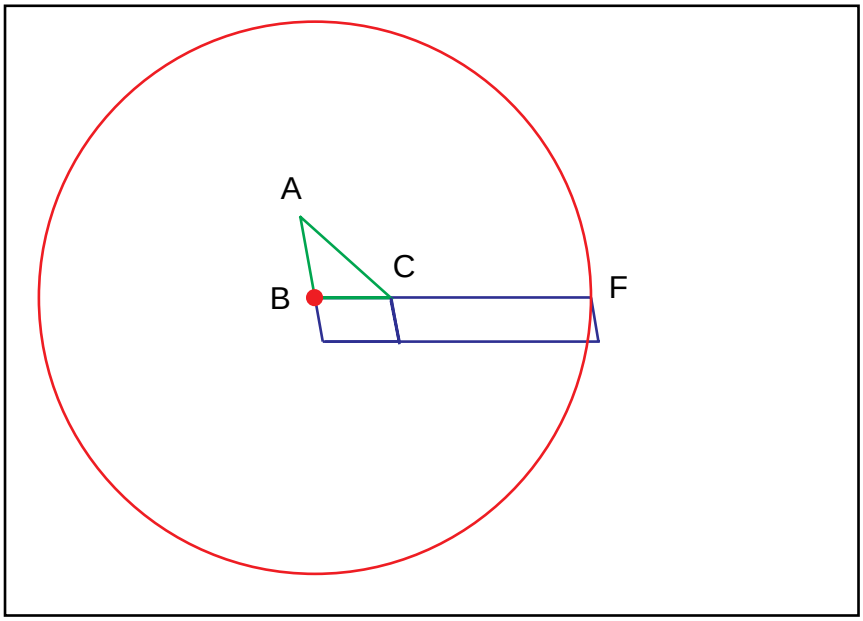
Stage 3.

VI.25:14. Now let GH be taken a mean proportional to BC, CF. [VI.13],

GOSUB VI.13, C#43.
VI.13:6. and let the semicircle BPF be described on BF;

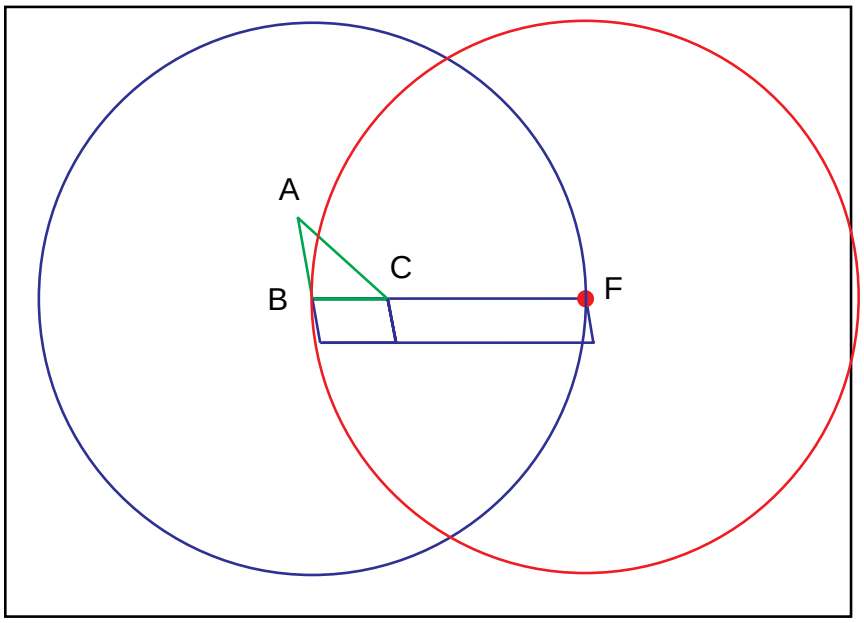


Bisect BF. [I.10]

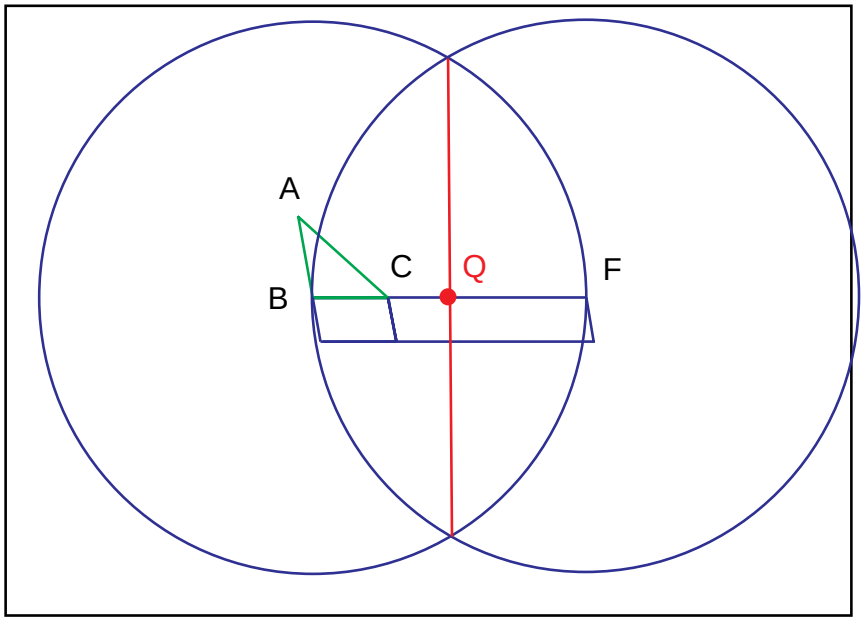


GOSUB I.10 (C#5).
Swing BF around B.

Swing FB around F.



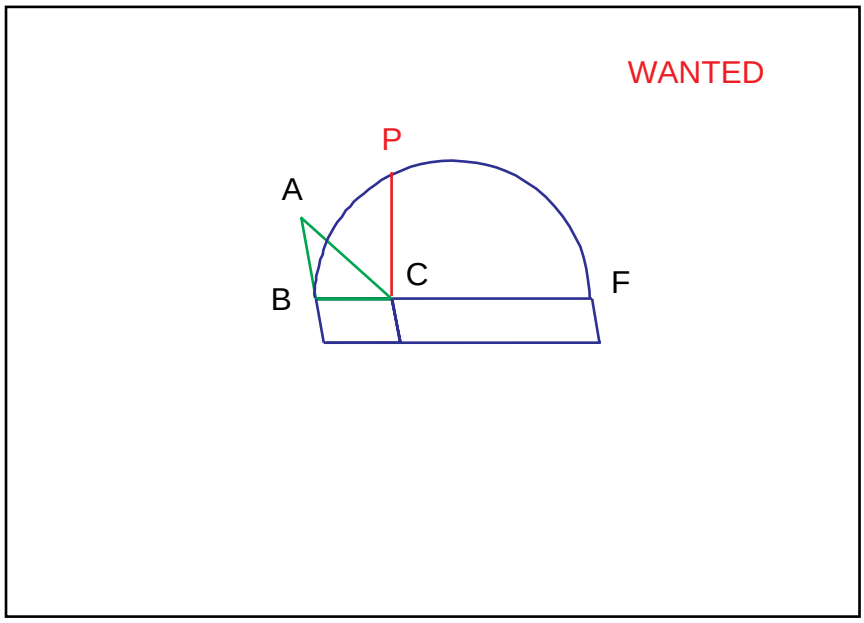
Connect the two crossing points and mark the midpoint Q of BF.



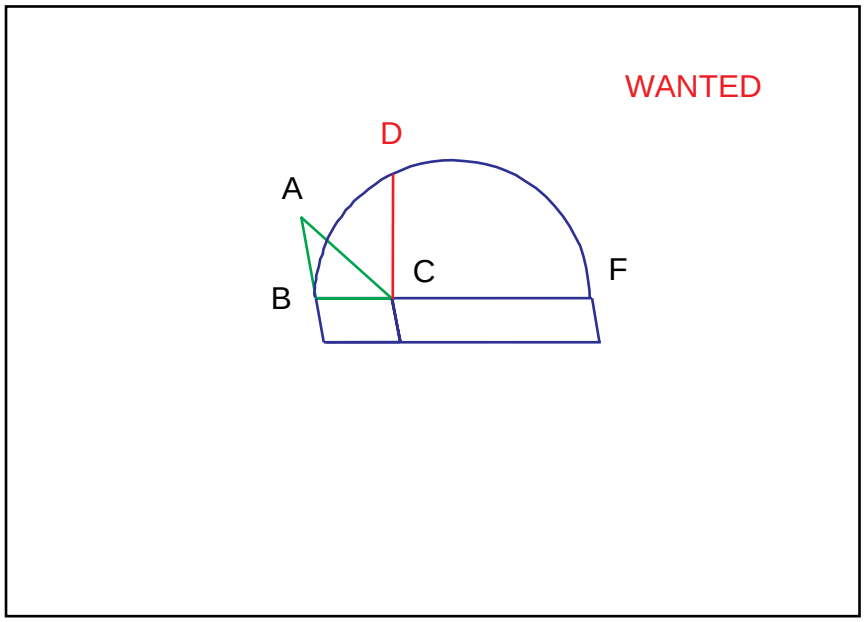
Cleanup. I.10.

RETURN to VI.13:6.

Swing BQ around Q.

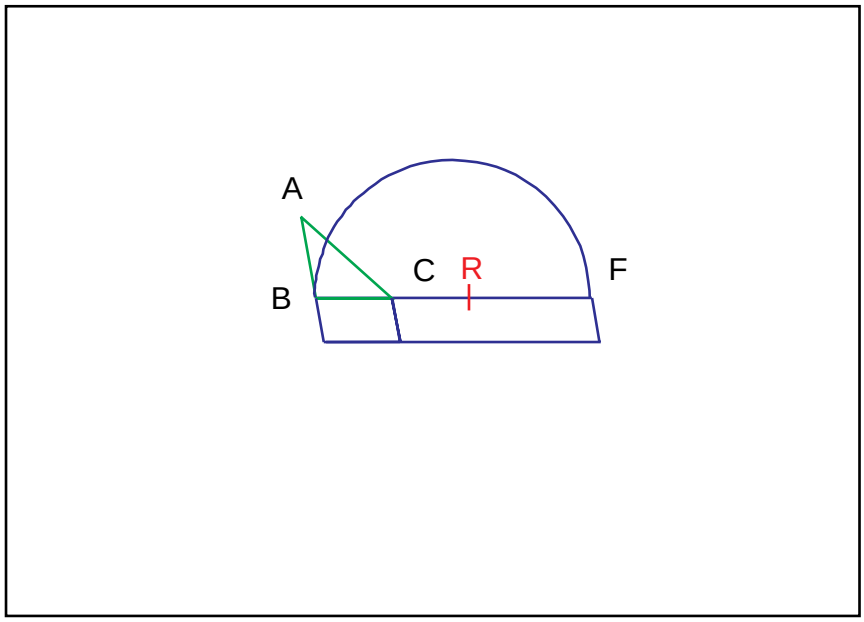


VI.13:8. let CP be drawn from the point C at right angles to the straight line BF.



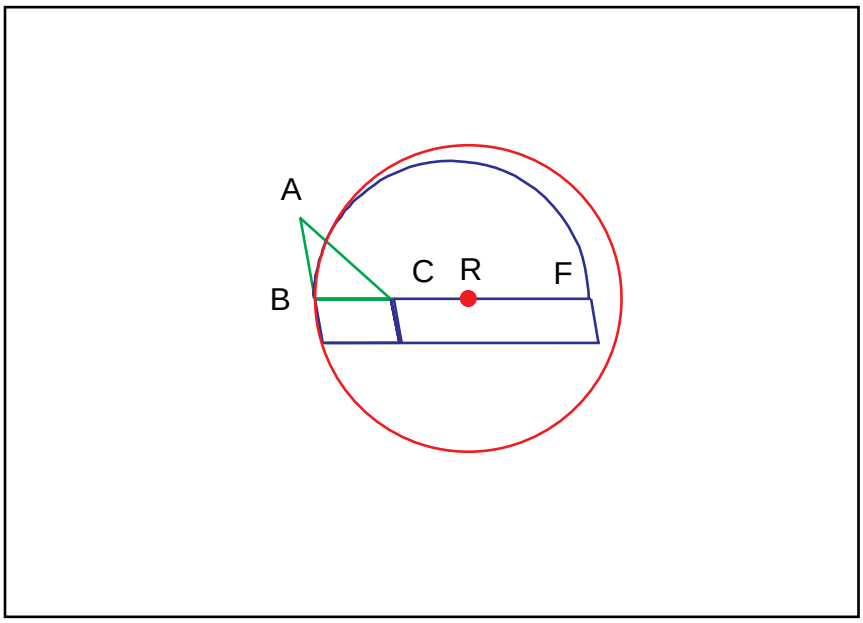
GOSUB I.11, C#6.
Retain current labels.

I.11:10. let CR be made equal to CB;

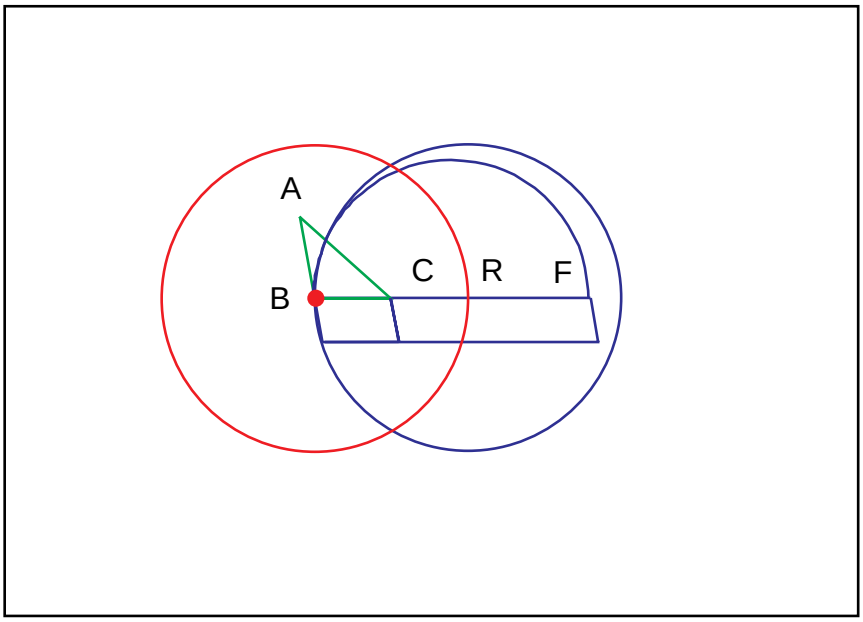


GOSUB I.1, C#1.

I.11:7. With centre R and distance RB let the circle ... be described;

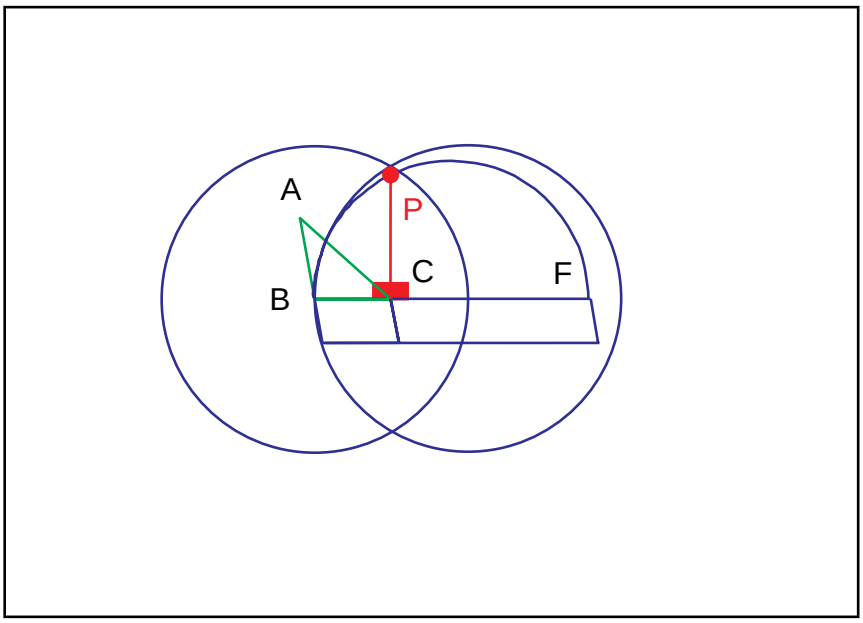


I.1:10. again, with centre B and distance BR let the circle ... be described;



RETURN to I.11:11.

Connect C to the upper crossing point and draw through to the point P on the semi-circle.

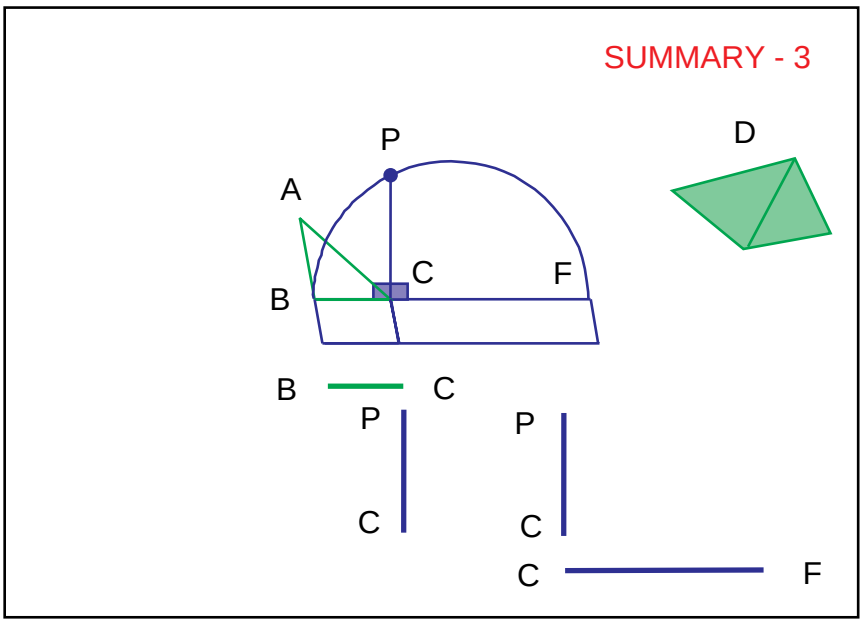


Cleanup.

RETURN to VI.13:8.

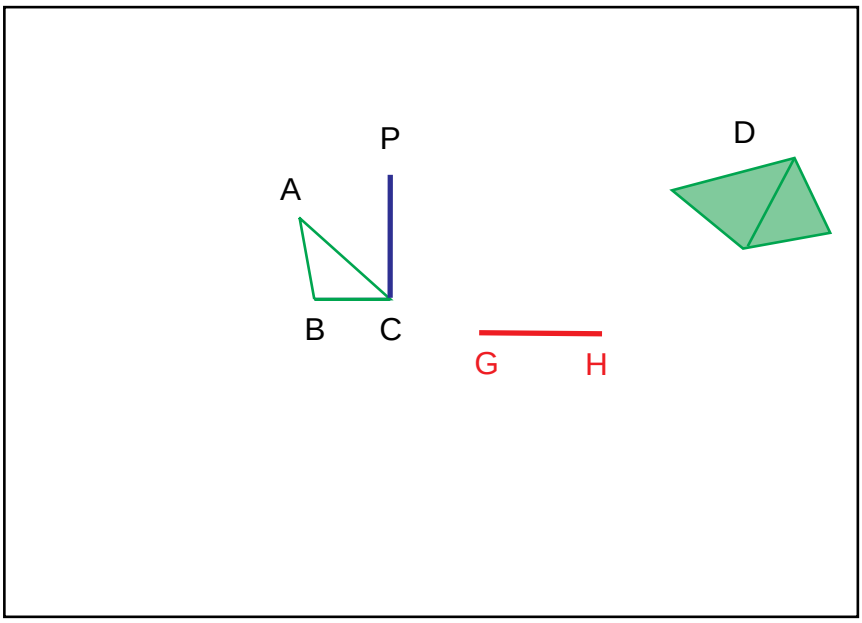
RETURN to VI.25:14.

Now state 3 is done. PC is the mean proportional between BC, CF.



Cleanup, preserve PC only.

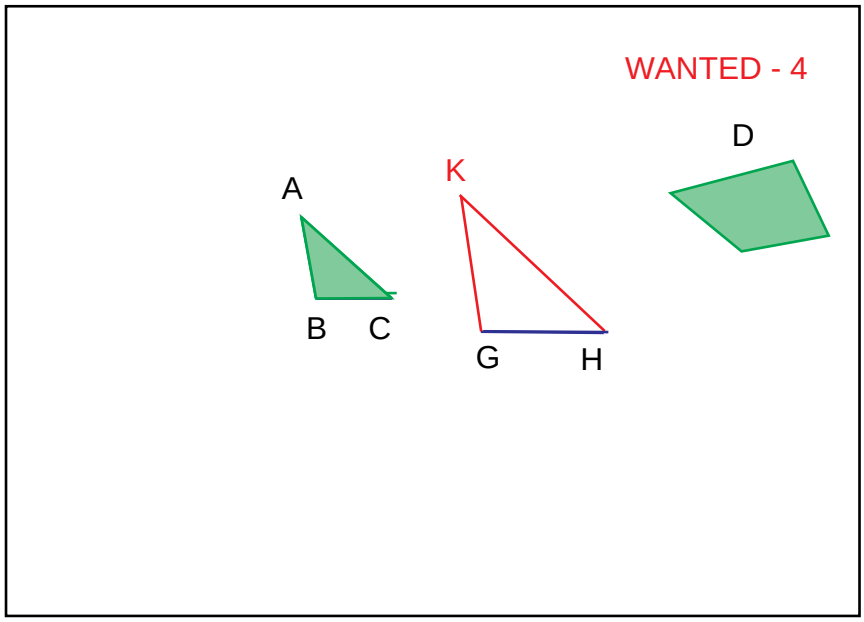
Move the mean proportional PC, to a horizontal line, GH, convenience.



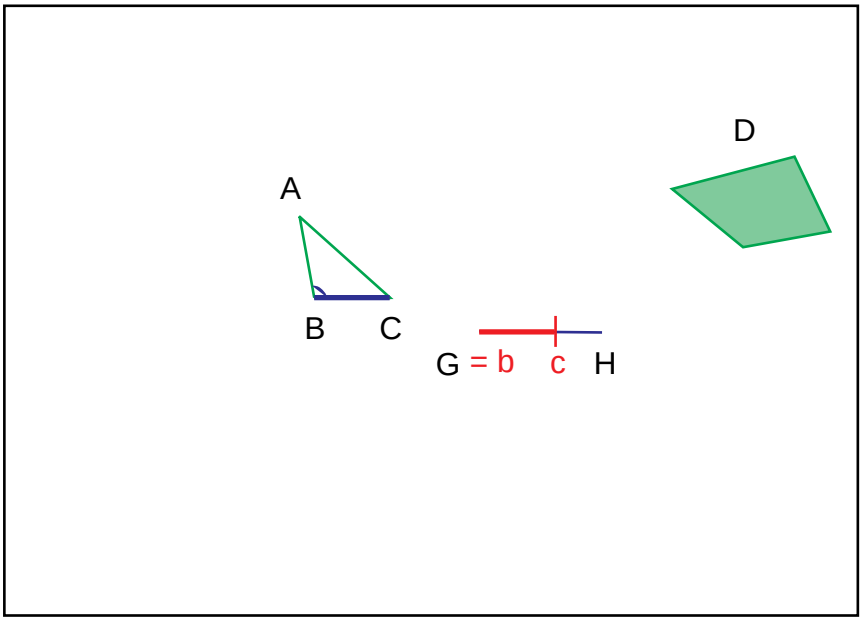
Stage 4.

VI.25:15. and on GH let KGH be described similar and similarly situated to ABC. [VI.18]

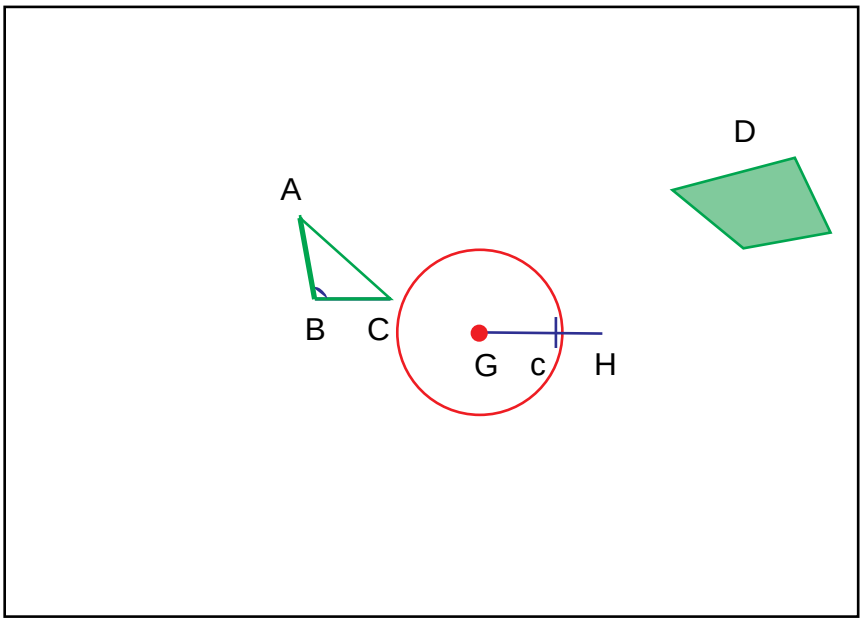
GOSUB VI.18, C#44.
(Steps 2 - 9 only.)



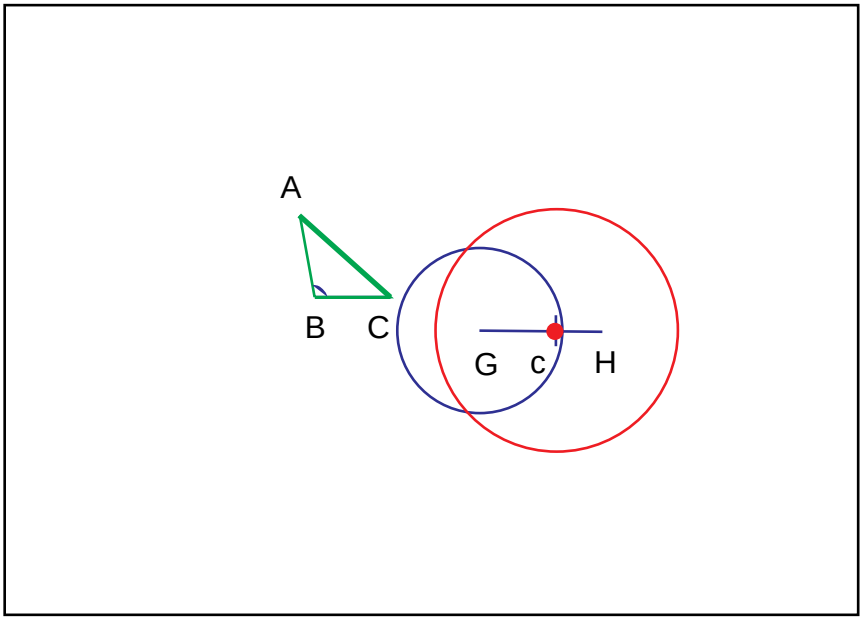
Move the base BC onto GH with the hot end b of the moved base bc at G.



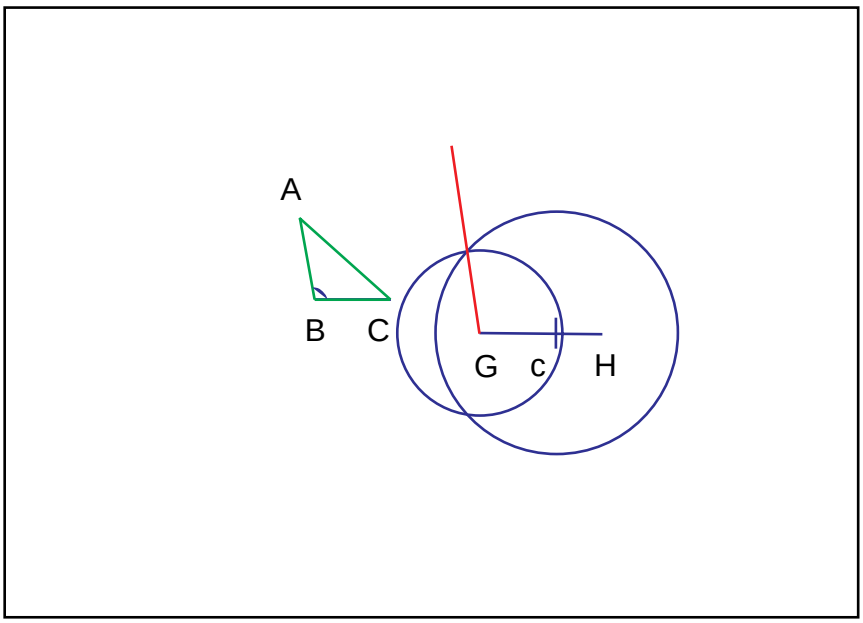
Swing the hot arm BA around the hot end b = G of the moved base.



Swing the cold arm CA around the cold end c of the moved base.

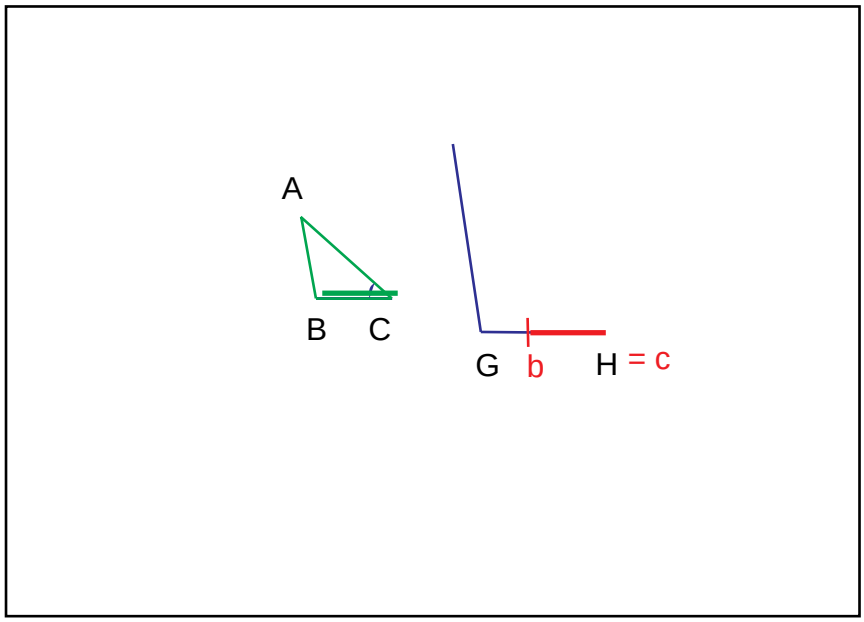


Connect the upper crossing point to G and extend upwards.

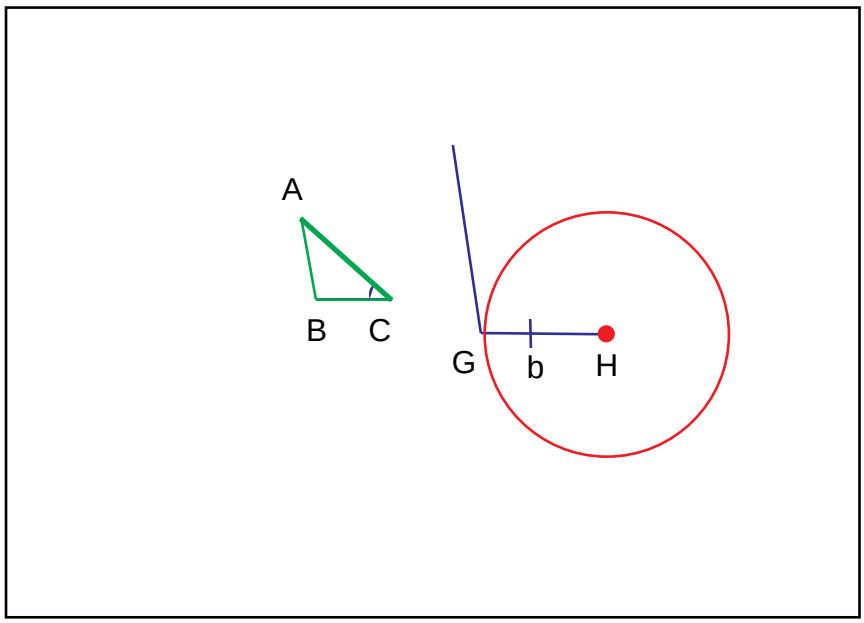


Cleanup.

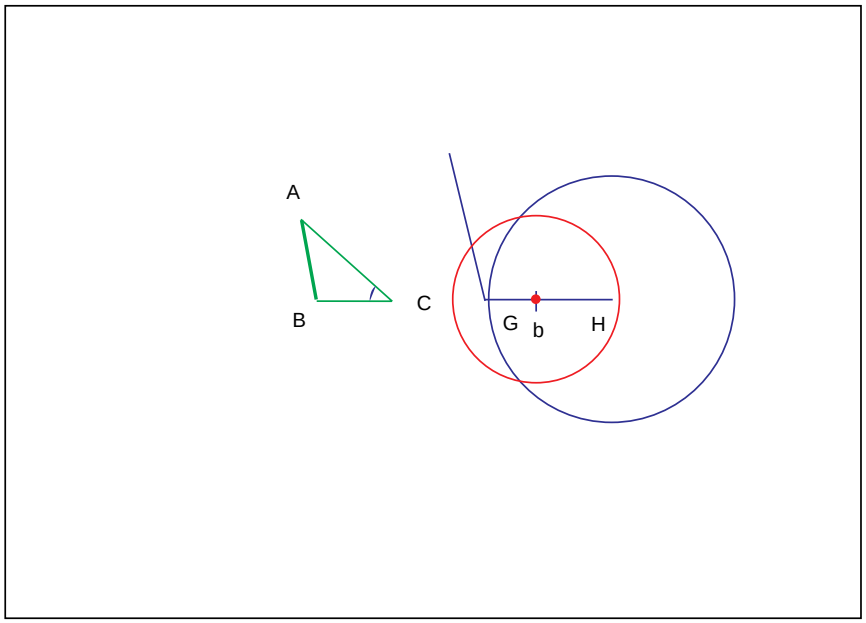
Move the base BC onto GH with the hot end c of the moved base at H.



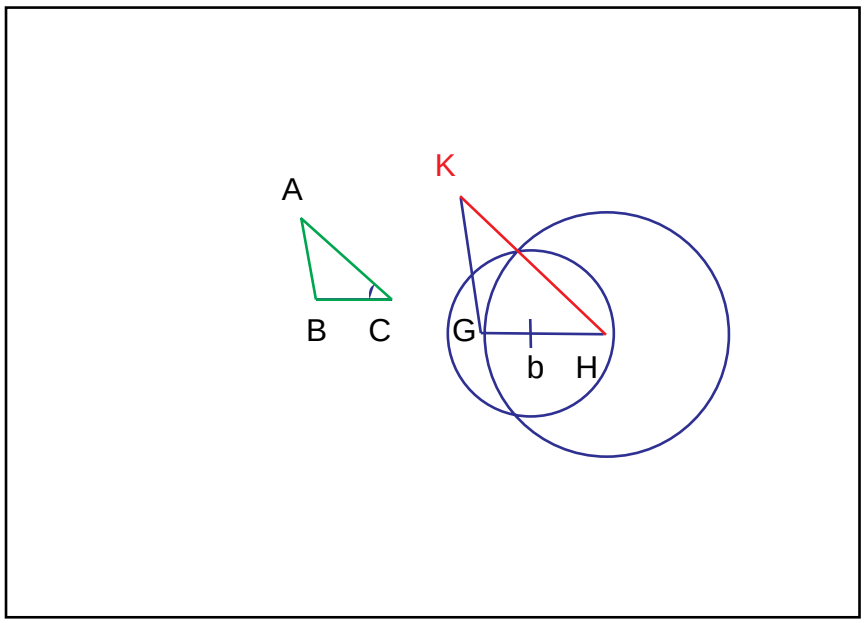
Swing the hot arm CA around the hot end c = H of the moved base.



Swing ther cold arm BA around the cold end b of the moved base.



Connect the upper crossing point to H and extend upward to meet the line from G at K.



Cleanup.
RETURN to VI.25.

DONE.

GHK has the size of D and the shape of ABC.